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Stochastic Assessment of Special-Rate Life Annuities

Annamaria Olivieri and Daniela Tabakova

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Abstract

Special-rate life annuities offer customized annuity rates, based on the lifestyle or health status of the individual. Their main purpose is to encourage the annuity demand, which is still underdeveloped in many markets; as better annuity rates are quoted for individuals showing a higher mortality profile, the number of individuals attracted by life annuities could increase. Providers should then gain larger pool sizes; however, this is possibly matched by a greater heterogeneity of the pool, due to several risk classes defined by the annuity design. Heterogeneity emerges not only in terms of different life expectancies, but also in respect of the dispersion of the lifetime distribution; indeed, situations resulting in a lower life expectancy also show greater variability of the lifetime. As it is well-known, pooling effects are reinforced by the pool size, while they are weakened by its heterogeneity, with a possibly unclear impact on the overall longevity risk to which the provider is exposed.

In this paper we investigate the longevity risk profile of an annuity pool consisting of several risk classes. We consider both the idiosyncratic and aggregate components of the risk, by modelling the random number of deaths and assuming a stochastic mortality dynamics. The heterogeneity of risk classes is represented alternatively in a deterministic and stochastic setting.

Our conclusions are in line with similar findings discussed in the literature, but obtained in a deterministic framework. Results suggest that the longevity risk profile of the provider is not significantly undermined by a greater pool heterogeneity, with a prevalence of the aggregate component whatever the pool composition.

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1 Introduction

Lower social security benefits and a general shift from defined benefit to defined contribution schemes in many pension systems increase the risk of ruin of individuals after retirement, as their personal wealth is more exposed than in the past to financial and longevity risk.

According to Yaari (1965), the best protection for the individual against the risk of ruin is offered by the (traditional) annuity product, which provides a lifetime income stream, independent of interest rates, the lifetime of the individual and the average lifetime of the population. However, annuity markets remain thin, as people are reluctant to buy annuities. Such a puzzle (the well-known annuity puzzle, see for example Benartzi et al. (2011)) has been largely analysed in the literature. Here, following Pitacco (2021), we note some features of the standard life annuity that could be considered undesirable by an individual, thus leading him/her to choices alternative to the life annuity. Because of the financial and longevity risk taken by the provider, annuities are priced with (possibly high) safety loadings and strict policy conditions. Considering the latter, for example, we recall that the annuity can neither be surrendered, nor the individual can apply for a change of the benefit amount. Thus, the money paid by the individual to purchase an annuity represents an illiquid and unredeemable investment, unable to adapt to the evolving needs of the annuitant. As a result, individuals in poor health or with poor lifestyles may refuse to underwrite an annuity, even if they need to supplement their post-retirement income. This favours adverse-selection effects, as individuals attracted by the product are those showing a higher life expectancy than the average. Providers account for such effects in annuity prices, which are then perceived even higher by many individuals.

Several innovations can be introduced in the annuity design. Those that can lead to lower prices include: a restriction of the payment period to older ages (resulting in a reduced number of payments); a linking of the benefit amount to a longevity experience (admitting future reductions of the benefit amount in case of unanticipated mortality improvements); the introduction of individual risk factors for pricing (resulting in higher annuity rates for those showing a lower life expectancy). We refer to Pitacco (2021) for an overview of these alternatives. In this paper we focus on the adoption of risk factors.

Annuities in which the pricing is based on risk factors are known as special-rate or underwritten annuities. The term "special-rate" comes from offering an annuity rate customized to the mortality profile of the individual, with better annuity rates if the lifestyle or health status suggest a higher probability of early death than the average. The specific status of the individual is checked via an appropriate underwriting step, which justifies the alternative expression "underwritten annuities". Special-rate life annuities have been adopted in some markets. They are described, for example, in Ainslie (2000), Drinkwater et al. (2006), Ridsdale (2012), Rinke (2002). Market issues, with particular regard to barriers, are discussed in Gatzert and Klotzki (2016), while Gracie and Makin (2006), James (2016) discuss practical aspects about pricing. Alternatives about the risk classification rules are addressed by Rinke (2002) and summarized in Pitacco (2021).

The pricing of special-rate life annuities requires the identification of a number of risk classes (or risk groups) when defining the tariff. A specific mortality assumption must then be adopted for each risk class. Poorer lifestyles or health conditions suggest a lower life expectancy, which is usually matched by a higher variance of the lifetime distribution, due to the variety of possible pathologies leading to a specific risk class, as well as to less data available than for standard annuities. In life insurance, differential mortality is usually represented in terms of additive or multiplicative adjustments to a baseline mortality assumptions. A review of the main models can be found, for example, in Olivieri (2006), Haberman and Olivieri (2014). A multiplicative adjustment coefficient is particularly common, but also convenient, as it can be extended quite nicely to a stochastic setting. This is, for instance, the underlying assumption of frailty models, defined by Vaupel et al. (1979).

Special-rate life annuities are designed to make the annuity more attractive and broaden the number of potential policyholders. Thus, a reasonable expectation for the provider is about an increase of the pool size. However, a higher degree of heterogeneity of the pool should also be envisaged, not only among risk classes, but also within classes. Indeed, risk classes grouping individuals with lower lifestyle standards or poorer health conditions will reasonably show a lower life expectancy but also a greater dispersion of the lifetime distribution. The impact of risk classification on the structure of the pool is addressed by Gatzert et al. (2012), Hoermann and Russ (2008), Olivieri and Pitacco (2016).

The life annuity business is exposed to financial and longevity risk. The latter, in particular, consists of an idiosyncratic component (due to random fluctuations in the number of deaths) and an aggregate component (due to the mortality dynamics). While the former is addressed by modelling the random number of deaths in a pool, in order to represent the latter it is necessary to adopt stochastic mortality rates. The literature suggest a number of approaches in this respect; reviews are provided by Cairns et al. (2008) and Hunt and Blake (2021). A key model is the Lee-Carter one (see Lee and Carter (1992)). Several authors have discussed either extensions to such a model or have introduced models with a similar structure; we mention, in particular, the proposal by Brouhns et al. (2002), largely adopted in actuarial practice.

Like any other insurance business, the management of an annuity pool is based on pooling arguments, which are related to mortality in the case of annuities. As it is well-known, the pooling effect is improved when dealing with larger sizes, but reduced when heterogeneity is greater. A research question then arises about the net impact of these two opposing effects on the overall risk profile of a pool of special-rate annuities. This problem has already been addressed by Pitacco and Tabakova (2022); their analysis suggests that the advantages gained because of a larger size outweigh the adverse effects of greater heterogeneity. The analysis performed by Pitacco and Tabakova (2022) is developed in a deterministic setting both in respect of heterogeneity and mortality dynamics. The purpose of this paper is to extend their framework first by introducing stochastic mortality rates, and then a stochastic level of heterogeneity within each risk class. We think that this way we can perform a more comprehensive analysis, as stochastic mortality rates allow us to account for the uncertain mortality dynamics, whereas stochastic assumptions with respect to heterogeneity allow us to address possible lack of knowledge about the risk classes composition. The results we obtain are anyhow in line with those in Pitacco and Tabakova (2022).

The remainder of the paper is arranged as follows. In Sect. 2, we describe the special-rate annuity products and the pool structures analysed in the paper. In Sect. 3 we describe the mortality and heterogeneity model. In Sect. 4 we define the actuarial quantities analysed in the numerical implementation, i.e. annuity rates and the present value of future benefits. In Sect. 5 we discuss the numerical implementation, while in Sect. 6 we conclude the paper with some summary and final remarks.

2 The products and the structure of the pool

2.1 The products and the related mortality assumptions

The specific labels given to the several types of special-rate annuities are not the same in all markets. Here, following Ridsdale (2012) and Pitacco (2021), we use the terminology common in the UK market.

A *standard life annuity* is priced assuming that the individual is in very good health and maintains an optimal lifestyle. The mortality assumption accounts for an adverse-selection effect, by adopting reduced mortality rates in respect of the general population. Further, mortality is projected to consider its dynamics over time, given the (long) temporal extent of the benefit.

A *lifestyle annuity* is priced considering risk factors related to smoking, drinking and eating habits, marital status, occupation, physical features such as height and weight, blood pressure and cholesterol levels. When compared to standard annuities, these are features that suggest a shorter life expectancy. While mortality is projected, mortality rates are assumed to be slightly higher than for standard annuities, resulting in a higher annuity rate.

An *enhanced life annuity* is priced based on the personal history of medical conditions, resulting in a reduced life expectancy. Mortality rates are projected, but they are assumed to be higher than for standard and lifestyle annuities.

An *impaired life annuity* is addressed to individuals whose medical conditions significantly shorten the expected lifetime (because, for example, of diabetes, chronic asthma, cancer, etc.). Given the mortality dynamics, mortality rates are assumed to be much higher than for standard annuities and higher than for enhanced annuities.

Finally, a *care annuity* is addressed to individuals with very serious impairments or in a senescent-disability (or long-term care) state. These products are typically classified as long-term care insurance. Because of the critical health situation of the individual, a lower life expectancy than for the other annuities is adopted.

Thus, moving from standard to care annuities, shorter life expectancies are assumed, resulting in higher annuity rates. Clearly, the provider can decide to offer just part of such products, based on commercial or other considerations.

The personal situation of the individual must be ascertained at issue, via a questionnaire (in particular, about his/her lifestyle), and a medical examination (in particular, when the individual suffers from critical illnesses). In life insurance, an individual showing extra-mortality is classified as a *substandard risk*; this is why impaired and care annuities are also called *substandard life annuities*.

2.2 The structure of the pool

We consider pools of annuities arranged into three classes: standard, enhanced and impaired life annuities. The label C_i is adopted to denote risk class i, where

- $i = 1$: Standard annuity;
- $i = 2$: Enhanced annuity;
- $i = 3$: Impaired annuity.

The pool is considered at an initial time 0, when annuities are issued. Whatever the risk class, annuitants are all age x_0 at policy issue; thus, they belong to the same cohort. We do not consider new entries after time 0; thus, the pool consists of a generation of policies.

The size at issue of risk class C_i is n_i, with $n = n_1 + n_2 + n_3$ the total pool size at time 0. The number of survivors in risk class C_i at time t, $t > 0$, is random, as a result of mortality. We use the notation $N_{t;i}$ for the number of survivors in risk class C_i at time t, while $N_t = N_{t;1} + N_{t;2} + N_{t;3}$ is the total number of survivors in the pool at the same time. We set $N_0 = n$ and $N_{0;i} = n_i$.

A lifetime fixed-amount annuity in arrears is paid to each annuitant, i.e. with payments at the end of each year. Each annuitant pays an initial capital S to the provider, and will get the annual amount b_i if placed in class C_i. The annuity rate $\frac{b_i}{S}$ is assessed as defined in Sect. 4. Here we just note that, due to the mortality assumptions, it will turn out $b_1 < b_2 < b_3$.

3 The mortality model

3.1 Baseline mortality

When modelling the lifetime of annuitants, whatever the type of annuity (either standard or special-rate), we need assumptions about the possible future mortality trend; further, when dealing with special-rate annuities we also need assumptions about the different mortality levels of the several risk classes, and the possible heterogeneity of each risk class.

As far as the mortality trend is concerned, we assume that all risk classes share a common mortality dynamics, which is measured on a reference population. We denote the relating mortality rates as $q_{x,t;\text{base}}$ and refer to them as the baseline mortality. As we comment below, the mortality rate in risk class C_i is obtained by adjusting the baseline mortality rate $q_{x,t;\text{base}}$.

We model $q_{x,t;\text{base}}$ by adopting the Poisson Lee-Carter model proposed by Brouhns et al. (2002). In such a model, the instantaneous force of mortality (in our case, for the reference population) is modelled as follows:

$$\mu_{x,t;\text{base}} = \exp(a_x + b_x \cdot k_t) \tag{3.1}$$

where a_x measures the average age-pattern of mortality over time, k_t describes the change in time in the level of mortality and b_x measures how such a change impacts the average mortality at age x. Errors in the representation of mortality are modelled by assuming a

Poisson distribution for the number of deaths, i.e.

$$D_{x,t;\text{base}} \sim \text{Poisson}(\text{ETR}_{x,t;\text{base}} \cdot \mu_{x,t;\text{base}}) \,, \tag{3.2}$$

where $\text{ETR}_{x,t;\text{base}}$ is the central number of exposed to risk in the reference population. Parameters are subjected to the constraints $\sum_t k_t = 0$, $\sum_x \beta_x = 1$. Parameter estimation requires stochastic simulation; we make use of the demography (see Hyndman (2023)) and StMoMo R packages (see Villegas et al. (2022)). The dataset we adopt is described in Sect. 5.1.

We note that (3.2) is based on the assumption of a piece-wise constant force of mortality; we then obtain the mortality rate as $q_{x,t;\text{base}} = \exp(-\mu_{x,t;\text{base}})$.

3.2 Mortality in the risk classes

The mortality rate for risk class C_i is assumed to be proportional to baseline mortality, with a proportionality coefficient that is lower or higher depending on the risk class. This is in line with many practical implementations of differential mortality; see, for example, Haberman and Olivieri (2014).

The proportionality coefficient can be assumed either deterministic or stochastic. The former choice is rather common when accounting for differential mortality due to observable risk factors. The latter, is at the basis of the frailty modelling (formally defined by Vaupel et al. (1979)), and represents an appropriate setting for unobservable risk factors.

In the paper, we test both choices. In both cases, the proportionality coefficient is assumed to be class-dependent, but age-independent. In a deterministic setting, we then assume that the mortality rate for risk class C_i is obtained as follows:

$$q_{x,t;i} = z_i \cdot q_{x,t;\text{base}} \,, \tag{3.3}$$

with $z_1 < z_2 < z_3$; in particular, $z_1 < 1$, considering the adverse-selection effect which is typical of standard annuities.

In a stochastic setting, the proportionality coefficient is random; we denote it as Z_i. Then, the mortality rate for risk class C_i is obtained as:

$$q_{x,t;i} = Z_i \cdot q_{x,t;\text{base}} \,. \tag{3.4}$$

A stochastic coefficient accounts, in particular, for the heterogeneity within the risk class, that is reasonably stronger for enhanced and impaired lives in respect of standard lives, due to the spectrum of diseases entitling to a common annuity rate. A stochastic coefficients also accounts for a greater uncertainty about the mortality level, originated by a lack of data about some groups of the population. It must also be noted that the underwriting process can miss or misunderstand some features of the risk. Some level of heterogeneity should then be expected in all risk classes (i.e., also in the class of standard risks).

We note that according to (3.4), the mortality rate of a risk class is affected by two sources of uncertainty: the mortality dynamics of the reference population and the heterogeneity in the risk class. Under the Lee-Carter model, a similar structure has been considered by Li et al. (2009) and Jarner (2021). Their purpose, however, is not to address the different heterogeneity levels of different risk groups, but to improve goodness-of-fit and the estimate of mortality improvements by accounting for the overall heterogeneity of the population (an aspect which is disregarded in the classical Lee-Carter model).

Similarly to the literature on frailty, and similarly to Li et al. (2009) and Jarner (2021), we assume for Z_i a Gamma probability distribution. Contrarily to frailty models, however, we assume fixed parameters for the probability distribution, i.e. we assume $Z_i \sim \text{Gamma}(\alpha_i, \beta_i)$ at all ages. Indeed, with Z_i we do not mean to measure the frailty of the population, but more simply the heterogeneity of the risk group, as it is defined by the underwriting process at issue. We recall that $\mathrm{E}[Z_i] = \frac{\alpha_i}{\beta_i}$ and $\mathrm{CV}[Z_i] = \frac{1}{\sqrt{\alpha_i}}$; thus, the ratio $\frac{\alpha_i}{\beta_i}$ between the parameters of the Gamma distribution describes the average level of differential mortality in risk class C_i with respect to the reference population, while parameter α_i describes the level of dispersion because of risk factors, namely the degree of heterogeneity within the risk class.

3.3 Modelling the number of deaths and the number of survivors

We assume that, conditional on the mortality dynamics (i.e., on the trajectory of the mortality rate over time), annuitants have independent lifetimes.

Under assumption (3.3), for any given trajectory of the mortality rates, that we denote as $q^{[j]}_{x,t;\text{base}}$, the number of deaths in year $(t-1, t)$ follows a binomial distribution:

$$D_{t-1;i} \sim \text{Bin}(N_{t-1;i}, z_i \cdot q^{[j]}_{x_0+t-1,t-1}) \,, \tag{3.5}$$

where $N_{t-1;i}$ must be replaced with the number of survivors observed at time $t-1$, and $N_{t;i} = N_{t-1;i} - D_{t-1;i}$.

Under assumption (3.4), first we approximate (3.5) with a Poisson distribution, i.e. we assume that (for each trajectory $q^{[j]}_{x_0+t-1,t-1}$ of baseline mortality) conditional on the realization z_i of the random coefficient Z_i, the number of deaths in year $(t-1, t)$ can be approximated with a Poisson distribution:

$$D_{t-1;i} \sim \text{Poisson}(N_{t-1;i} \cdot z_i \cdot q^{[j]}_{x_0+t-1,t-1}) \,. \tag{3.6}$$

Such an approximation is widely adopted in the actuarial literature (see, for example, Gerber (2013), Panjer and Willmot (1992)). Once we assign a Gamma(α_i, β_i) distribution to Z_i, we can easily find that the unconditional distribution of the number of deaths follows a Negative Binomial distribution (see, again, Gerber (2013), Panjer and Willmot (1992)):

$$D_{t-1;i} \sim \text{NegBin}\left(\alpha_i, \frac{\beta_i/(N_{t-1;i}, z_i \cdot q^{[j]}_{x_0+t-1,t-1})}{1+\beta_i/(N_{t-1;i}, z_i \cdot q^{[j]}_{x_0+t-1,t-1})}\right) , \tag{3.7}$$

where α_i, β_i are the parameters of the Gamma distribution for Z_i (for the computations of the parameters of the Negative Binomial distribution, see also Olivieri and Pitacco (2016)). We note that in (3.7), as in (3.5), $N_{t-1;i}$ must be replaced with the number of survivors observed at time $t-1$, having $N_{t;i} = N_{t-1;i} - D_{t-1;i}$.

4 Annuity rates and present value of future benefits

We aim at investigating the impact of the pool size and composition on the probability distribution of the total payout of the pool, with particular regard to its dispersion.

We first define the present value at time 0 of future benefits for risk class C_i as follows:

$$\text{PVFB}_{0;i} = \sum_{t=1}^{\omega-x_0} b_i \cdot N_{t;i} \cdot v(t) , \tag{4.1}$$

where $v(t)$ is the discount factor and ω is the maximum attainable age (and, therefore, $\omega - x_0$ is the maximum attainable payment duration). We restrict our attention to longevity risk, and introduce a deterministic financial setting. Financial risk is clearly an important matter in the risk management of an annuity pool; however, it is not impacted by the pool composition. In order to have a better and more immediate understanding of the results affected by longevity risk, we then prefer to disregard financial risk. This way, $\text{PVFB}_{0;i}$ is random because the numbers of survivors $N_{t;i}$'s are random, as it is described in Sect. 3.3. As far as the maximum attainable age ω is concerned, we assume a fixed value, as it is common in many actuarial applications.

The present value at time 0 of future benefits for the whole pool is:

$$\text{PVFB}_0 = \sum_i \text{PVFB}_{0;i} . \tag{4.2}$$

The annuity rate $\frac{S}{b_i}$ for risk class C_i is obtained in a traditional way, i.e. as the actuarial value at time 0 of future benefits, using best-estimate assumptions:

$$\frac{S}{b_i} = \sum_{t=1}^{\omega-x_0} {}_t p^{[\text{BE}]}_{x_0;i} \cdot v(t) , \tag{4.3}$$

where ${}_tp^{[\mathrm{BE}]}_{x_0;i}$ is obtained from the best-estimate trajectory of baseline mortality rates (that we assume to correspond to the median of their probability distribution), multiplied by z_i under assumption (3.3) and by $\mathrm{E}[Z_i]$ under assumption (3.4). We note that in the actuarial literature an annuity rate obtained as in (4.3) is usually denoted as a_{x_0} (symbol that we do not use explicitly, having used a_x to represent a parameter of the Lee-Carter model; see (3.1)). We also note that by using best-estimate assumptions no premium loading is included in (4.3), contrarily to what is usual in the traditional actuarial pricing model.

In order to analyse the risk profile of the pool, we assess the ε quantiles of the present value at time 0 of the future benefits for the whole pool, PVFP_0, that we denote as $\mathrm{PVFP}_0^{\varepsilon}$. We consider alternative pool sizes and compositions. The difference $\mathrm{PVFP}_0^{\varepsilon} - n \cdot S$ (where $n \cdot S$ represents the total amount of money cashed by the provider from annuitants at time 0) can be interpreted as an additional amount required in consideration of the risks to which the pool is exposed. In our setting, risks are due to mortality and their magnitude is affected by the pool size and composition. It is a provider's decision how to cover this additional amount, in particular how much with premium loading to be charged to annuitants and how much with equity. This choice is not discussed in this paper.

5 Implementation: Results and discussion

5.1 Parameters

We consider individuals age $x_0 = 65$ at time 0, born in year 1958. Baseline mortality is calibrated on Italian data extracted from the Human Mortality Database. Mortality of the male population is considered. To avoid the impact of major random fluctuations at the oldest ages, the maximum attainable age is set $\omega = 100$ years. The Poisson Lee-Carter model described in Sect. 3.1 is calibrated on calendar years $(1960, 2019)$ (this choice of the time-frame has provided better fit than alternative periods). Projections are then obtained until the assumed life limit of the cohort. As already mentioned, for calibration and projection we use the demography (by Hyndman (2023)) and StMoMo R packages (by Villegas et al. (2022)). We identify the best-estimate mortality as the median of the projected mortality rates. According to the best-estimate trajectory, the remaining expected lifetime (at age 65) of the cohort in the reference population is 22.975 years.

Mortality of standard, enhanced and impaired risks is obtained as a proportion of baseline mortality, as we have discussed in Sect. 3.2. The level of mortality in the various classes depends, in particular, on the risk factors chosen by the provider for underwriting purposes. We assume that mortality of standard risks is lower than baseline mortality; this is in line with empirical evidence in many markets. Conversely, we assume that enhanced and impaired risks have higher mortality levels than baseline mortality; this is consistent with the features of such risks. As to the coefficients z_i's in a deterministic setting and the parameters of the Gamma distribution for coefficients Z_i's in a stochastic setting, we make the assumptions summarized in Table 1, in terms of the fixed value of deterministic coefficients z_i's, and the expected value and coefficient of variation of the stochastic coefficients Z_i's.

Table 1: Parameters describing the mortality of risk classes in relation to baseline mortality.

Risk class C_i	Deterministic coefficients z_i	Stochastic coefficients $\mathrm{E}[Z_i]$	$\mathrm{CV}[Z_i]$	Remaining expected lifetime (best-estimate scenario)
C_1: Standard	0.7	0.7	3.16%	25.529
C_2: Enhanced	2	2	8.16%	17.907
C_3: Impaired	4	4	11.18%	13.146

First, we note that the parameters of the Gamma distribution are chosen so that the expected value of the coefficients Z_i's is the same as the fixed value of the corresponding deterministic coefficients z_i's. This way, the expected lifetime for each group in the best-estimate scenario is the same under a deterministic and stochastic heterogeneity framework. Life expectancies for the various risk classes are listed in the last column of Table 1. We also note that an assumption of a mortality level for standard risk which is on average 70% of baseline mortality is in line with similar (average) assumptions adopted in market tables for standard annuities (for example, in the Italian market). Assumptions about enhanced and impaired lives imply life expectancies whose difference in respect of the life expectancy of standard risks is in line with what assumed by Pitacco and Tabakova (2022). When we consider stochastic coefficients, we assume some heterogeneity also in the class of standard risks. This is reasonable, as pools in practice are never perfectly homogeneous. The level of heterogeneity for enhanced and impaired lives is assumed to be higher, due to the spectrum of illnesses leading to such classes (as well as to some uncertainty originated by fewer data available).

As for the pool size and composition, we test some of the cases considered in Pitacco and Tabakova (2022). Indeed, as we stated in Sect. 1, we aim at extending the analysis discussed in such paper, by considering a stochastic mortality and heterogeneity framework. Table 2 lists the pools whose risk profile is analysed in the numerical implementation.

Table 2: Pool structures.

Pool label	Size			
	Class C_1 Standard	Class C_2 Enhanced	Class C_3 Impaired	Pool
P01	10 000	0	0	10 000
P02	10 000	100	0	10 100
P03	10 000	500	0	10 500
P04	10 000	1 000	0	11 000
P05	10 000	0	100	10 100
P06	10 000	0	500	10 500
P07	10 000	0	1 000	11 000
P08	10 000	500	250	10 750
P09	10 000	1 000	500	11 500
P10	9 750	500	250	10 500
P11	9 500	1 000	500	11 000

Pool P01 only consists of standard annuities; thus, it is a traditional annuity pool, and we use it as a reference for interpreting in comparative terms the risk profiles of pools also including other risk classes. Pools P02–P04 also include enhanced annuities, with a different size of such a risk class. Similarly, pools P05–P07 also include impaired annuities, in addition to standard annuities. Considering pools including, apart from standard annuities, only enhanced or only impaired annuities allows us to interpret more clearly the impact on the pool's total riskiness of some types of risks. Pools P08–P11 include all risk classes, with different possible sizes of each risk class. In particular, pools P10–P11 take into account possible "cannibalization effects", i.e. a reduction of the number of standard annuities, emerging if some annuitants willing to underwrite a standard annuity are classified as enhanced or impaired life or if they simply decide not to take the annuity, when they compare prices of the different risk classes. In all cases, the standard annuity risk class is the largest in the portfolio, as it is reasonable (and as it is assumed by Pitacco and Tabakova (2022)).

As to the discount factor, we set a flat 2% discount rate. The initial capital paid by each annuitant is set to $S = 1\,000$ monetary units (in the following, to make it short, when quoting amounts we will leave out the reference to "monetary units"). Table 3 quotes the amount of the annuity benefit for each risk class, obtained using annuity rates (4.3).

Table 3: Annuity benefit amount for an initial capital $S = 1\,000$.

Risk class C_i	Benefit amount b_i
C_1: Standard	52.49
C_2: Enhanced	70.85
C_3: Impaired	93.49

Clearly, the different benefit amounts are a result of the different assumptions about the mortality levels in each class; see, in particular, the values of the life expectancy quoted in Table 1.

5.2 Risk profile of the pool: Numerical results and discussion

Tables 4 and 5 quote the ε quantiles of the present value at time 0 of future benefits for the whole pool, as a % of the total initial capital, i.e. $n \cdot S$, assessed in different situations regarding the components of longevity risk and the heterogeneity of the risk classes, as we will comment later. Here we comment on how we have chosen ε. We focus on the right tail of the probability distribution of PVFB_0, and then we consider high values for ε, namely $\varepsilon = 0.9$ and $\varepsilon = 0.995$. Chosen ε, higher values of $\frac{\text{PVFB}_0^{\varepsilon}}{n \cdot S}$ express greater exposure to adverse fluctuations for the provider. The value $1 - \varepsilon = 0.005$ can be referred to as an acceptable level of default probability. In this view, $\text{PVFB}_0^{0.995}$ represents the total amount of resources the provider needs to hold in order to cover, with probability 0.5%, its obligations. As already noted, it is a decision of the provider how to fund such resources, in particular how much via premiums and how much with equity. The quantity $\text{PVFB}_0^{0.9}$ could, for example, represent the amount to be covered with the premiums (by charging to annuitants and additional amount on top of S or by reducing the annuity rate so that the total initial capital corresponds to $\text{PVFB}_0^{0.9}$). Then, $\text{PVFB}_0^{0.995} - \text{PVFB}_0^{0.9}$ would be covered with equity. In this interpretation, the higher $\frac{\text{PVFB}_0^{0.9}}{n \cdot S}$, the higher the required premium loading. Conversely, the higher the difference $\text{PVFB}_0^{0.995} - \text{PVFB}_0^{0.9}$, the higher the required equity.

Table 4 quotes $\text{PVFB}_0^{0.9}$ and $\text{PVFB}_0^{0.995}$ as a % of the total initial capital amount $n \cdot S$. We consider fixed coefficients for differential mortality, i.e. model (3.3).

In case a, we only consider the best-estimate of mortality rates, whereas the number of deaths is modelled (and simulated) with the Binomial distribution. We can then interpret

Table 4: ε quantiles of PVFB_0, as a percentage of the total initial capital $n \cdot S$: $\frac{\mathrm{PVFB}_0^{\varepsilon}}{n \cdot S} \cdot 100$. Fixed coefficient for differential mortality.

Pool		Case a		Case b		Case c	
Label	Size	$\varepsilon = 0.9$	$\varepsilon = 0.995$	$\varepsilon = 0.9$	$\varepsilon = 0.995$	$\varepsilon = 0.9$	$\varepsilon = 0.995$
P01	10 000	100.395%	100.786%	102.159%	104.160%	102.188%	104.233%
P02	10 100	100.396%	100.798%	102.172%	104.194%	102.202%	104.263%
P03	10 500	100.394%	100.780%	102.221%	104.317%	102.253%	104.373%
P04	11 000	100.388%	100.761%	102.279%	104.445%	102.314%	104.487%
P05	10 100	100.398%	100.788%	102.182%	104.221%	102.211%	104.280%
P06	10 500	100.399%	100.804%	102.266%	104.405%	102.300%	104.462%
P07	11 000	100.408%	100.791%	102.364%	104.604%	102.400%	104.677%
P08	10 750	100.398%	100.786%	102.272%	104.428%	102.308%	104.480%
P09	11 500	100.392%	100.755%	102.373%	104.627%	102.410%	104.691%
P10	10 500	100.387%	100.816%	102.275%	104.433%	102.315%	104.478%
P11	11 000	100.400%	100.807%	102.383%	104.646%	102.423%	104.716%

Case a: Deterministic baseline mortality rates (best-estimate trajectory); Binomial distribution for the number of deaths.

Case b: Stochastic baseline mortality rates; for each trajectory, number of deaths equal to the expected value for each trajectory.

Case c: Stochastic baseline mortality rates; for each trajectory, Binomial distribution for the number of deaths.

the values of quantiles as being affected only by the random fluctuations of the number of deaths. When there are several risk classes, there is a bit more dispersion (as it emerges, for example, if we compare the quantiles of pool P04 with pool P11: these are two pools of the same size, but of different composition). A trade-off between pool size and heterogeneity emerges in many situations. For example, quantiles for pool P02 are higher than pool P01 (which is smaller), while quantiles for pool P03 are lower than pool P01. On the other hand, pools P05–P07 show increasing quantile values, despite the increase of the pool size. Overall, however, the change of the value of quantiles appears to be little.

In case b, we consider stochastic baseline mortality rates, whereas for the number of deaths we take them to be the same as their expected value in each trajectory. Thus, we can interpret the values of the quantiles as being affected only by systematic fluctuations, originated by the uncertain mortality dynamics. Comparing these quantiles to the corresponding ones in case a, higher values emerge, in particular at the very right tail (i.e., when we consider higher values for ε). This witnesses the higher impact of aggregate than idiosyncratic longevity risk. The effect of heterogeneity emerges when comparing the quantiles obtained for different pools. It is useful to point out that higher values of mortality rates imply a larger dispersion of the lifetime distribution. Thus, even if at this stage we are only taking a deterministic description of heterogeneity, the distribution of the lifetime of enhanced and impaired lives is anyhow more dispersed than for standard lives, due to the higher level of mortality rates.

In case c, we both consider stochastic baseline mortality rates and, for each trajectory, the number of deaths is modelled (and simulated) with the Binomial distribution. We point out that in this case we perform a two step simulation: one step is for the baseline mortality rate and, nested on that, one step is for the number of deaths. In this case, we can interpret the values of quantiles as being affected both by random fluctuations and systematic deviations. Clearly, the values taken by quantiles are the highest among the three cases. Interpretations are in line with what commented for cases a and b.

In Table 5 we consider random coefficients for differential mortality, i.e. model (3.4).

In case d, again we only consider the best-estimate of mortality rates, but this time the number of deaths is modelled (and simulated) with the Negative Binomial distribution, which accounts for the heterogeneity in the various risk classes. Again, quantiles only capture random fluctuations, that impact a little bit more than case a in Table 4 as a consequennce of greater heterogeneity. A trade-off between pool size and composition emerges, but overall quantiles look very similar for the various pools.

Case e includes both random fluctuations and systematic deviations (again, we perform a two step simulation). If we compare this case to case c in Table 4, we find higher values for the quantiles, but with slight differences.

We point out that Table 5 does not include a situation with only systematic deviations, as (given the choice of parameters) we would find what depicted by case b in Table 4.

As we have noted, the level of heterogeneity of each risk class is in particular a consequence of the risk factors adopted by the provider for the underwriting process. It is interesting to test an alternative situation about the level of heterogeneity, that could result from adopting different underwriting rules in respect of those underlying the summary statistics for the Z_i's reported in Table 1. More specifically, we now consider parameters for the coefficients Z_i's describing a greater heterogeneity than what considered before. The assumptions are summarized in Table 6 in terms of expected value and coefficient of variation of the Z_i's.

Table 7 quotes the quantiles of the present value at time 0 of future benefits for the whole pool obtained with this alternative set of parameters.

Table 5: ε quantiles of PVFB_0, as a percentage of the total initial capital $n \cdot S$: $\frac{\text{PVFB}_0^{\varepsilon}}{n \cdot S} \cdot 100$. Random coefficient for differential mortality.

Pool		Case d		Case e	
Label	Size	$\varepsilon = 0.9$	$\varepsilon = 0.995$	$\varepsilon = 0.9$	$\varepsilon = 0.995$
P01	10 000	100.421%	100.840%	102.196%	104.239%
P02	10 100	100.422%	100.829%	102.209%	104.267%
P03	10 500	100.420%	100.834%	102.261%	104.370%
P04	11 000	100.420%	100.858%	102.320%	104.494%
P05	10 100	100.420%	100.854%	102.219%	104.291%
P06	10 500	100.441%	100.850%	102.307%	104.471%
P07	11 000	100.455%	100.886%	102.411%	104.690%
P08	10 750	100.426%	100.855%	102.314%	104.482%
P09	11 500	100.429%	100.852%	102.417%	104.693%
P10	10 500	100.435%	100.869%	102.318%	104.488%
P11	11 000	100.441%	100.888%	102.431%	104.722%

Case d: Deterministic baseline mortality rates (best-estimate trajectory); Negative Binomial distribution for the number of deaths.

Case e: Stochastic baseline mortality rates; for each trajectory, Negative Binomial distribution for the number of deaths.

We compare Table 7 to Table 5. The greater assumed heterogeneity clearly implies higher values for the quantiles, but the change is not dramatic. Other comments are in line with those already discussed before.

6 Conclusions

Due to ongoing changes in the area of post-retirement benefits, caused by the demographic dynamics and new risks, great efforts are dedicated to finding new solutions that can reconcile individuals' needs with providers' requirements. Indeed, it is likely that individuals and providers have opposing viewpoints. Restricting our attention to life annuities, we note that several innovations have been addressed in the literature, including topics such as rider benefits, restrictions to the payment period, linking the annuity benefit to a longevity experience and customizing prices to the specific risk profile of the individual.

Table 6: Alternative parameters for the stochastic coefficients Z_i's.

Risk class C_i	Stochastic coefficients $\mathrm{E}[Z_i]$	$\mathrm{CV}[Z_i]$
C_1: Standard	0.7	5%
C_2: Enhanced	2	10%
C_3: Impaired	4	15.81%

In this paper, we focus on the latter topic, considering the so-called special-rate annuities.

When offering special-rate annuities, the target of the provider is to increase the pool size. Indeed, individuals in poor health or with poor lifestyle may reject standard annuities, as they are priced with reference to healthy individuals with very good lifestyles. Individuals in poor health or with poor lifestyle could instead accept annuities rated considering their features, i.e. higher mortality. The availability of special-rate annuities should then favour the increase of the demand for annuities.

In terms of the risk management of the pool, a larger size is usually considered to be advantageous, as it improves pooling effects, while greater heterogeneity (which is introduced when structuring the pool in several risk classes) is considered to be disadvantageous, as it increases the dispersion of the provider's total payout.

An analysis of these two opposing effects has already been considered in Pitacco and Tabakova (2022). The aim of this paper is to extend their investigation, by addressing a stochastic mortality dynamics and stochastic levels of heterogeneity in the risk classes. In their conclusions, Pitacco and Tabakova (2022) actually mentioned stochastic mortality as an interesting extension of their work.

As far as the trade-off between size and heterogeneity is concerned, our findings are in line with those of Pitacco and Tabakova (2022): heterogeneity worsen the risk profile of the provider's obligations, but this is usually mitigated by a larger size. Special-rate annuities, then, can really represent a way for the provider to extend the annuity business keeping exposure to risk at levels similar to less heterogeneous pools. As an addition to the findings in Pitacco and Tabakova (2022), our conclusion is that, above all, the aggregate longevity risk proves to be the most significant source of risk for the provider, and in relative terms it is little affected both from the pool size and heterogeneity. Because of its systematic nature, its management actually requires innovative actuarial principles and market solutions. The research in this area is very rich, as it is well documented by Blake

Table 7: ε quantiles of PVFB_0, as a percentage of the total initial capital $n \cdot S$: $\frac{\mathrm{PVFB}_0^{\varepsilon}}{n \cdot S} \cdot 100$. Random coefficient for differential mortality, with parameters expressing greater heterogeneity.

Pool		Case f		Case g	
Label	Size	$\varepsilon = 0.9$	$\varepsilon = 0.995$	$\varepsilon = 0.9$	$\varepsilon = 0.995$
P01	10 000	100.474%	100.954%	102.207%	104.255%
P02	10 100	100.471%	100.938%	102.219%	104.278%
P03	10 500	100.464%	100.935%	102.269%	104.383%
P04	11 000	100.466%	100.914%	102.329%	104.505%
P05	10 100	100.471%	100.954%	102.229%	104.299%
P06	10 500	100.484%	100.993%	102.317%	104.487%
P07	11 000	100.515%	100.996%	102.423%	104.710%
P08	10 750	100.473%	100.945%	102.321%	104.491%
P09	11 500	100.470%	100.961%	102.426%	104.710%
P10	10 500	100.467%	100.903%	102.328%	104.501%
P11	11 000	100.489%	100.978%	102.439%	104.727%

Case f: Deterministic baseline mortality rates (best-estimate trajectory); Negative Binomial distribution for the number of deaths, with alternative parameters.

Case g: Stochastic baseline mortality rates; for each trajectory, Negative Binomial distribution for the number of deaths, with alternative parameters.

et al. (2023). It is worth noting that, similarly to Pitacco and Tabakova (2022), we have disregarded financial risk, being concerned about longevity risk. However, financial risk is clearly an important risk source in an annuity pool, and requires appropriate decisions when designing the overall risk management strategy of the provider.

Declarations

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Dedication: This paper is dedicated to prof. Ermanno Pitacco.

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