ARC Centre of Excellence in Population Ageing Research

Working Paper 2024/03

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Hélène Morsomme, Jennifer Alonso-García and Pierre Devolder

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March 15, 2024

Abstract

Population ageing undermines traditional social security pension systems that combine pay-as-you-go (PAYG) and defined benefits (DB). Indeed, demographic risk, if guaranteed benefits remain unaltered, will be borne entirely by workers through increases in the contribution rate. To avoid a substantial increase of the contributions and in order to maintain simultaneously the financial sustainability and the social adequacy of the public pension system, risk sharing and automatic balancing mechanisms need to be put in place. We present a two-step convex family of risk-sharing mechanisms. The first shares the risk between contributors and retirees through adjustments in the contribution rate, used to calculate the global covered wage bill, and the benefit ratio that represents the relationship between average pensions and wages. The second step studies how the retirees' risk should be shared between the different retirees' generations through adjustments in the replacement rate and a sustainability factor that affects pension indexation during retirement. We perform a detailed study of the effect of social planner's targets and solidarity weight between various generations in a deterministic and stochastic environment.

Keywords: risk-sharing, automatic balancing mechanisms, pension design, ageing
JEL: H55, J18, G22

* Earlier drafts of this paper were circulated under the title: "Public pension schemes: intergenerational risk sharing" and "Intergenerational risk sharing in public pension schemes". The authors are responsible for any errors. Correspondence to: Jennifer Alonso García, Université Libre de Bruxelles, Department of Mathematics, Faculté des Sciences, Campus de la Plaine - CP 213, Boulevard du Triomphe ACC.2, 1050 Bruxelles, Belgium.

*Corresponding author: jennifer.alonso.garcia@ulb.be .

1 Introduction

Population is ageing at a global scale. Life expectancy and fertility rate are expected to further increase and decrease, respectively (United Nations, 2022). As a result of this, the share of global population at ages 65 and above is projected to rise from 10 to 16 percent by 2050. In Europe and Northern America it is even expected to rise to over 60 percent by 2050 (United Nations, 2023). Obviously, the increase in old-age to working-age ratio, that is, the ratio of people over 65 over working age population, inevitably stresses the financial sustainability of pay-as-you-go (PAYG) financed public pension schemes. As a consequence, most countries enforce pension reforms that entail adjustments in retirement ages, benefits and contributions (OCDE, 2023).

Some of these reforms introduce Risk-sharing mechanisms (RSMs), often known as Automatic Adjustment Mechanisms (AAMs) or Automatic Balancing Mechanisms (ABMs) to restore sustainability. These RSMs are a set of *predetermined* measures setablished by law to be applied immediately according to indicators that reflect the financial health of the system (Vidal-Meliá et al., 2009). Indeed, about two-thirds of OECD countries have at least one RSM in place (OCDE, 2021). Mechanisms include those embodied in notional DC schemes (e.g. Sweden (Settergren, 2013)), links to the statutory retirement age to life expectancy (as done in Finland and Portugal (Carmen Boado-Penas et al., 2020)) and benefit adjustments to changes in demographic ratios or the wage bill (as done in Canada or Germany (Börsch-Supan and Wilke, 2004)). However, to the best of the authors knowledge, the literature justifying the particular RSM architecture is thin.

Intergenerational risk sharing has also gained interest in academic research. Generally, the literature focuses on studying the sustainability of the PAYG pension scheme for a pre-specified pension setting and subsequently apply RSMs to restore sustainability (Alonso-García et al., 2018b; Alonso-García and Devolder, 2019).[1] For instance Knell (2010) analyzes the internal rate of return of generic pension schemes after incorporating adjustment mechanisms and find that the length of retirement period relative to the length of the working period should be included in the adjustment parameter design whereas Godínez-Olivares et al. (2016a) and Godínez-Olivares et al. (2016b) use nonlinear dynamic programming to study ABMs strategies that vary the contribution rate, retirement age and indexation of pensions to achieve financial health. More recently Alonso-García et al. (2018a) study tractable RSMs that involve changes in the contribution rate to the pension system and indexation rate of existing pensions to restore the sustainability of both Defined Benefit (DB) and Defined Contribution (DC) pension schemes.

Our proposal, akin to what Holzmann et al. (2013) and Alonso-García and Devolder (2019) do in a DC context, addresses the RSM question differently using a two step approach. In the first step, we study which parameters can the social planner afford, as in which *benefit ratio* – the ratio of global pensions to average wages – and *contribution rate* – the part of the wages that is paid to the PAYG system –, given the PAYG sustainability constraint. This philosophy aligns with the concept of hybrid pension schemes, as an intermediate solution between a pure DB and DC, pioneered by Musgrave (1981) which has recently gained scholar attention (Devolder and de Valeriola, 2019; Schokkaert et al., 2020; Torricelli, 2022). The idea presented in Musgrave (1981) is that the contribution and benefit ratio should follow a

[1] An increasing interest in RSMs is present also in funded pension schemes, see for instance Cui et al. (2011); Donnelly (2017) and Chen et al. (2023). However, the methodology differs from that used in pay-as-you-go given the nature of the risks on scope.

rational joint evolution. We achieve this goal through a quadratic loss function (Cairns, 2000) whereby the optimal parameters are affected by a long-term target and demographic weights. We call this the *1st* level of risk-sharing.

In a second stage, we study how the risk related to benefit ratio changes should be shared between the different retirees' generations through adjustments in the *replacement rate* – the factor used to calculate the pension for new retirees –, and a RSM that we denote the *sustainability factor* that affects pre-specified pension indexation levels during retirement. The latter has a structure that resembles the sustainability factor studied by Baurin and Hindriks (2023) which is implemented in Germany (Börsch-Supan and Wilke, 2004). However, compared to existing RSMs in place, we provide the scientific foundation to develop a flexible hybrid family of PAYG pension systems based on a double RSM that can be seen as intermediate solutions between DB, where all risk is borne by contributors, and DC, where all risk is borne by retirees.

The remainder of this paper is structured as follows. Section 2 presents the general stylized pay-as-you-go pension system in which the RSM is studied. Section 3 presents basic RSMs as intermediate solutions between pure DB and pure DC. We assess their suitability in terms of risk sharing between workers and retirees in Subsection 3.1 or risk sharing among retirees in Subsection 3.2. Optimal solutions in a deterministic context are presented in Section 4. We devise the two levels of risk-sharing families and obtain the optimal parametrization for chosen long-term social planner's goals. We generalize in Section 5 our solutions to a context where the dependency ratio is stochastic and present not only central estimates but also its associated uncertainty and Section 6 concludes.

2 General model

This section presents the general stylized pay-as-you-go pension sytem in which the risk sharing scheme is studied. We develop a general discrete model with time and age-dependent parameters for which the risk sharing will be studied.

The principle of a pay-as-you-go (PAYG) system is that the current contributions cover the current pension benefits. It means that at each moment, the global contributions should equal the global benefits. In this work we denote the total income from contributions C_t as

$$C_t = \pi_t \, S_t \, W_t, \tag{1}$$

with π_t the contribution rate, S_t the mean salary and W_t the number of workers at time t. The total pension expenditure is then given by

$$E_t = \overline{P}_t \, R_t = \delta_t \, S_t \, R_t, \tag{2}$$

with $\overline{P}_t$ the mean pension benefits, R_t the number of retirees and δ_t the benefit ratio representing the mean pension benefits over the mean salary. In this setting retirement age is constant and the same for everyone[2]. Similarly, we assume that the individual and mean salary coincide. Then, the first individual pension benefit when retiring at age x_r time t is

$$P_{x_r,t} = \widetilde{\delta}_t \, S_t, \tag{3}$$

[2]We focus on a design with constant retirement age as we aim to compare risk sharing schemes on a like-for-like basis. Indeed, linking retirement age to life expectancy has a limited effect on DC pension schemes since pensions rely on actual life expectancy whereas it can be a tool to increase sustainability for DB schemes under the assumption that the labour market absorbs the older workers fully (Alonso-García et al., 2018a).

where S_t is the t value of the last salary and $\widetilde{\delta}_t$ is remplacement rate for a generation retiring at time t. Unlike the cohort-specific replacement rate $\widetilde{\delta}$, the benefit ratio δ is a cross-sectional measure on all retirees. During the retirement period, we assume that pensions are indexed following the evolution of average wages adjusted by a sustainability factor as

$$P_{x_r+\Delta,t+\Delta} = P_{x_r,t}\,\frac{S_{t+\Delta}}{S_t}\,\beta_t^{\Delta} \tag{4}$$

where β_t^{Δ} is the sustainability factor between retirement t and calculation $t+\Delta$. By convention, the sustainability factor for a duration of 0 is equal to 1, that is, $\beta_t^0 = 1$. As earlier stated, PAYG rely on an equilibrium between income from contribution and pension expenditures:

$$C_t = E_t, \tag{5}$$

which, with a minor rewrite, corresponds to

$$\pi_t = \frac{R_t}{W_t}\delta_t = D_t\,\delta_t \tag{6}$$

with D_t the dependency ratio — the ratio between the number of retirees R_t to the number of workers W_t. This dependency ratio is the risk factor representing the demographic trend.

Remark 1 (Cross-sectional benefit ratio δ_t)*.* The cross-sectional benefit ratio δ represents, as earlier stated, the weighted average of the cohort-specific replacement rates $\widetilde{\delta}$ reduced by the sustainability factors β for all retirees. The corresponding equilibrium equation corresponds to

$$\delta_t = \sum_{x=x_r}^{\omega} \widetilde{\delta}_{t-(x-x_r)}\,\beta_{t-(x-x_r)}^{x-x_r}\,l_{x,t}. \tag{7}$$

where $l_{x,t} = \frac{L_{x,t}}{R_t}$ is the retired population density function with $L_{x,t}$ the total population aged x at time t and R_t the total number of retirees at time t.

Proof: The total pension expenditure is calculated as the sum of all pensions paid in the system as follows:

$$E_t = \sum_{x=x_r}^{\omega} L_{x,t}P_{x,t} = \sum_{x=x_r}^{\omega} L_{x,t}P_{x_r,t-(x-x_r)}\frac{S_t}{S_{t-(x-x_r)}}\beta_{t-(x-x_r)}^{x-x_r},$$

where $L_{x,t}$ represents the number of individuals aged x at time t. Since the initial and indexed pension are given by Equations (3) and (4) respectively, we obtain:

$$\begin{aligned} E_t &= \sum_{x=x_r}^{\omega} L_{x,t}P_{x_r,t-(x-x_r)}\frac{S_t}{S_{t-(x-x_r)}}\beta_{t-(x-x_r)}^{x-x_r} \\ &= \sum_{x=x_r}^{\omega} L_{x,t}\tilde{\delta}_{t-(x-x_r)}S_{t-(x-x_r)}\frac{S_t}{S_{t-(x-x_r)}}\beta_{t-(x-x_r)}^{x-x_r}. \end{aligned}$$

Since the total number of retirees is given by $R_t = \sum_{x=x_r}^{\omega} L_{x,t}$ we can rewrite the expression above as follows:

$$E_t = S_t\,R_t \sum_{x=x_r}^{\omega} \frac{L_{x,t}}{R_t}\,\tilde{\delta}_{t-(x-x_r)}\,\beta_{t-(x-x_r)}^{x-x_r}.$$

Defining the retired population density as

$$l_{x,t} = \frac{L_{x,t}}{R_t}, \tag{8}$$

we obtain the expression (7) above. Note of course that l is well defined as a density since $l \geq 0 \; \forall x, t$ and $\sum_{x=x_r}^{\omega} l_{x,t} = 1$. $\square$

Replacing (7) in (6) yields

$$\pi_t = D_t \sum_{x=x_r}^{\omega} \widetilde{\delta}_{t-(x-x_r)} \, \beta_{t-(x-x_r)}^{x-x_r} \, l_{x,t} \, . \tag{9}$$

The expression (9) highlights the relationship between the contribution rate π, the replacement rates $\tilde{\delta}$ and sustainability sustainability factors β. The risk sharing mechanisms outlined in Sections 3 and 4 propose adjustments of these parameters according to demographic risk D_t.

3 Risk sharing mechanisms: two levels

In this section we analyze classical pension schemes and risk-sharing mechanisms and discuss how they fare in terms of adequacy. We assess their suitability in terms of risk sharing between workers and retirees in Subsection 3.1 and risk sharing among retirees in Subsection 3.2.

3.1 First level: risk sharing between the workers and the retirees

Here we study three basic risk-sharing mechanisms: pure defined benefit (DB), pure defined contribution (DC) and the so-called Musgrave rule that adapts either the contribution rate π_t and/or the benefit ratio δ_t depending on the mechanism. These parameters are adapted to satify (5). For instance, in an ageing context, an increasing dependency ratio[3] D implies either an increase of the contribution rate π, a decrease of the benefit ratio δ, or both.

Pure DB

In pure DB pension schemes, the level of the real pension benefit is constant. This has the natural implication that pensions for new retirees will be equivalent to those of older generations. Hence, the replacement rate used to calculate the first pension needs to be constant $\tilde{\delta}_{t-x+x_r} = \tilde{\delta}_t$ and indexation cannot divert from the wage-increase which implies that the sustainability factor should be equal to 1 ($\beta_s^i = 1 \; \forall i, s$):

$$\begin{aligned} P_{x,t} &= P_{x_r,t-x+x_r} \frac{S_t}{S_{t-x+x_r}} \overbrace{\beta_{t-x+x_r}^{x-x_r}}^{=1} = \tilde{\delta}_{t-x+x_r} S_{t-x+x_r} \frac{S_t}{S_{t-x+x_r}} = \tilde{\delta}_{t-x+x_r} S_t = \tilde{\delta} \, S_t, \\ P_{x_r,t} &= \tilde{\delta}_t \, S_t = \tilde{\delta} \, S_t. \end{aligned}$$

In that case, the benefit ratio δ_t (7) is constant and simplifies to $\delta_t = \tilde{\delta} = \delta$. To maintain sustainability (5) for varying D_t, the only degree of freedom left is the contribution rate π_t:

$$\pi_t = D_t \cdot \delta. \tag{10}$$

[3] In case of decrease of the dependency ratio D, the presented mechanisms are exactly the same with opposite variations of the adjustment variables.

A change in the contribution rate impacts the workers through the level of their contributions. The demographic risk is entirely borne by the workers' contributions which in counterpart will receive a fixed target level of real pension benefits.

Pure DC

On the contrary, for the pure DC system, the level of the contribution is constant, that is, $\pi_t = \pi$. The benefit ratio δ_t is adapted according to the change of the dependence ratio D_t

$$\delta_t = \frac{\pi}{D_t} \, . \tag{11}$$

A change in the benefit ratio impacts the retirees through the level of their benefits. Obviously, the benefit ratio δ_t (7) is an aggregate figure that is dependent on the level of current and past replacement rates $\tilde{\delta}$ and sustainability factors β_s^i. In a second stage, it would be necessary to assess how the global change in δ_t stemming from the risk-sharing mechanism (11) translates into the different generations. This will be addressed in detail in Subsection 3.2. In any case, in this scenario, the demographic is entirely borne by the retirees' benefits.

Musgrave

Ideally, the contribution rate π and benefit ratio δ should be adjusted simultaneously based on (6). A possible intermediate between the DB and DC systems is the Musgrave rule (see e.g. Musgrave (1981) and Schokkaert et al. (2020)). The idea is that the net replacement rate denoted as M – the ratio between the pension and the salary net of contributions – is constant:

$$M = \frac{\overline{P}_t}{\overline{S}_t \, (1 - \pi_t)} = \frac{\delta_t}{1 - \pi_t} \, . \tag{12}$$

Remark 2 (Benefit ratio δ and contribution rate π under the Musgrave rule). We can obtain δ_t and π_t under the Musgrave rule by exploiting the cross-sectional PAYG equilibrium (6). By doing so, we guarantee that the system is both *fair* under the *Musgrave* philosophy and sustainable from a basic pure PAYG financing perspective.

Rewriting (12), we get

$$M = \frac{\delta_t}{1 - \pi_t} \Leftrightarrow M(1 - \pi_t) = \delta_t \overbrace{\Leftrightarrow}^{(6)} M(1 - \pi_t) = \frac{\pi_t}{D_t} \Leftrightarrow M = \pi_t(M + \frac{1}{D_t}),$$

which yields

$$\pi_t = \frac{M \, D_t}{1 + M \, D_t}. \tag{13}$$

Using a similar rationale for δ_t, we obtain:

$$\delta_t = \frac{M}{1 + M \, D_t}. \tag{14}$$

□

The Musgrave rule can be interpreted as the same variation for the pension benefits and for the salaries net of contributions

$$M_t = M_{t+1} = M \Leftrightarrow \frac{\overline{P}_t}{S_t\left(1-\pi_t\right)} = \frac{\overline{P}_{t+1}}{S_{t+1}\left(1-\pi_{t+1}\right)} \Leftrightarrow \frac{\overline{P}_{t+1}}{\overline{P}_t} = \frac{S_{t+1}\left(1-\pi_{t+1}\right)}{S_t\left(1-\pi_t\right)} \,. \tag{15}$$

The Musgrave rule can be seen as a plausible intermediate between the extreme DB and DC systems since both the contribution rate π_t and benefit ratio δ_t are susceptible of change under an adverse demographic scenario. However, any change is joint as to maintain a net replacement rate M. Of course, the Musgrave rule is just one of an array of other possible intermediate systems. Therefore, in Section 4 we develop a family of risk sharing mechanisms that we will then compare to the pure DB, DC and Musgrave rule presented here.

3.2 Second level: risk sharing between retirees

Once the first level risk sharing is determined, the second level focuses on the retired generations. It determines how the change of pension expenditures is shared between newly retired cohorts by adjusting their replacement rate $\widetilde{\delta}$ and older retirees through the sustainability factor β. These parameters are adapted as to satify the pension expenditure equilibrium through Equation (7). As in the previous level, we can consider various scenarios. First, we consider the scenario where old pensions are fully wage-indexed ($\beta = 1$), and finally the case where all retirees are affected in the same way ($\tilde{\delta}_t = \delta_t$).

Stable pension for old retirees

For the first extreme, pension benefits are totally indexed according to the wage evolution. It corresponds to a sustainability factor equal to 1:

$$\beta = 1 \,.$$

In this case, Equation (7) becomes:

$$\delta_t = \sum_{x=x_r}^{\omega} \widetilde{\delta}_{t-(x-x_r)}\, l_{x,t}, \tag{16}$$

which, highlighting the difference between new retirees $l_{x_r,t}$ at time t and old retirees, can be rewritten as

$$\begin{aligned} \delta_t &= \tilde{\delta}_t l_{x_r,t} + \sum_{x=x_r+1}^{\omega} \widetilde{\delta}_{t-(x-x_r)}\, l_{x,t}, \\ \tilde{\delta}_t &= \frac{\delta_t - \sum_{x=x_r+1}^{\omega} \widetilde{\delta}_{t-(x-x_r)}\, l_{x,t}}{l_{x_r,t}}. \end{aligned} \tag{17}$$

In a scenario with an increasing dependency ratio D_t, the replacement rate $\widetilde{\delta}$ is adjusted based on the reduced value of the benefit ratio δ (eq. 7), determined through the first-level risk-sharing mechanism. In this scenario, the level of indexation for the older generation remains unaltered despite potential costs to the PAYG system. A change in the replacement rate impacts solely the newly retired generation. The demographic risk within the retiree population is entirely borne by new retirees, who adjust their initial pension benefits.

Solidarity between all retirees

A natural counterpart considers the same level of adjustment for new and old retirees by fixing the replacement rate to the benefit ratio:

$$\tilde{\delta}_t = \delta_t \tag{18}$$

In that case, the corresponding sustainability factor that satisfies the benefit ratio equilibrium (7) becomes:

$$\delta_t = \sum_{x=x_r}^{\omega} \delta_{t-(x-x_r)}\, \beta_{t-(x-x_r)}^{x-x_r}\, l_{x,t} \Leftrightarrow \delta_t = \delta_t\, l_{x_r,t} + \sum_{x=x_r+1}^{\omega} \delta_{t-(x-x_r)}\, \beta_{t-(x-x_r)}^{x-x_r}\, l_{x,t}$$

$$\Leftrightarrow 1 = l_{x_r,t} + \sum_{x=x_r+1}^{\omega} \frac{\delta_{t-(x-x_r)}}{\delta_t}\, \beta_{t-(x-x_r)}^{x-x_r}\, l_{x,t} \Leftrightarrow \sum_{x=x_r+1}^{\omega} l_{x,t} = \sum_{x=x_r+1}^{\omega} \frac{\delta_{t-(x-x_r)}}{\delta_t}\, \beta_{t-(x-x_r)}^{x-x_r}\, l_{x,t}.$$

Of course, for $x > x_r$, when

$$\frac{\delta_{t-(x-x_r)}}{\delta_t}\, \beta_{t-(x-x_r)}^{x-x_r} = 1,$$

or alternatively

$$\beta_{t-(x-x_r)}^{x-x_r} = \frac{\delta_t}{\delta_{t-(x-x_r)}},$$

we obtain the sought equilibrium. The constraints in this scenario then become

$$\begin{cases} \tilde{\delta}_t &= \delta_t \\ \beta_t^{\Delta} &= \dfrac{\delta_{t+\Delta}}{\delta_t} \end{cases}. \tag{19}$$

The constraints given in (19) imply that the risk is balanced accross all retirees. Someone who retired in t, sees their pension reduce in the course of $\Delta = x - x_r$ by β_t^{Δ} compared to the base wage indexation:

$$\frac{P_{x,t+\Delta}}{P_{x_r,t}} = \frac{\tilde{\delta}_t S_t \frac{S_{t+\Delta}}{S_t} \beta_t^{\Delta}}{\tilde{\delta}_t S_t} = \frac{S_{t+\Delta}}{S_t} \beta_t^{\Delta} = \frac{S_{t+\Delta}}{S_t} \frac{\delta_{t+\Delta}}{\delta_t}.$$

For newly retirees, the expression is given by:

$$\frac{P_{x_r,t+\Delta}}{P_{x_r,t}} = \frac{\tilde{\delta}_{t+\Delta}\, S_{t+\Delta}}{\tilde{\delta}_t\, S_t} = \frac{S_{t+\Delta}}{S_t} \frac{\delta_{t+\Delta}}{\delta_t}.$$

The evolution of pensions for all cohorts hence coincides.

In Section 4 we present a family of risk sharing mechanisms between these two extreme scenarios.

4 Risk sharing mechanisms: optimal choice

Section 3 presented various *basic* risk-sharing options commonly used in practice or the literature. However, it is unclear to what extent these choices are made optimally. Hence, this section proposes an optimal risk-sharing mechanism that is decomposed into two levels. Firstly, we optimize the contribution rate π of the workers and the benefit ratio δ of the retirees in Section 4.1. In a second stage, we analyze in Section 4.2 how the adaptation of the benefit ratio generated by the first step can be shared among different generations of retirees by optimizing the replacement rate $\widetilde{\delta}$ of new retirees and the sustainability factor β affecting indexation of pensions paid to curent retirees.

4.1 First level: risk sharing between the workers and the retirees

We propose a family of risk-sharing mechanisms that satisfy the pure PAYG equilibrium (6) based on the optimization of a quadratic loss function as in Cairns (2000). Our objective function measures the joint stability of the contribution rate π and the benefit ratio δ around fixed target values respectively $\overline{\pi}$ and $\overline{\delta}$ as in Godínez-Olivares et al. (2016a,b). Relative deviation from these target values is punished. We suppose a weight on each component ρ: the relative deviation of the contribution rate π and the relative deviation of the benefit ratio δ. This weight parameter characterizes the family of intermediate systems between the DB and DC systems. Note that the weight parameter ρ is a choice that reflects the preferences of the social planner, akin to the choice of the risk aversion parameter within an utility framework. We let ρ_t be general and time-dependent as it could be used as a tool to study a transition from pure DB to DC or viceversa.

We consider a quadratic objective function[4] where the part linked to the contribution rate is weighted by the number of individuals of working age W_t and the part linked to the benefit ratio affecting retirees is weighted by the total number of retirees R_t. This is a proxy to the voting weight every generation would have in negotiating the proposed pension reforms (Cetin and Hindriks, 2023; Baurin and Hindriks, 2023). The objective function to be minimised is then given by:

$$f(\rho_t, \delta_t, \pi_t) \quad = \quad \rho_t\, W_t \left(\frac{\pi_t}{\overline{\pi}} - 1\right)^2 + (1-\rho_t)\, R_t \left(\frac{\delta_t}{\overline{\delta}} - 1\right)^2 . \tag{20}$$

Since the number of workers $W_t > 0$, the optimum value obtained through expression (20) is equivalent to the optimum obtained through $\frac{f(\rho_t,\delta_t,\pi_t)}{W_t}$. Rewriting it in the latter manner allows to only deal with one demographic factor, namely $D_t = \frac{R_t}{W_t}$ as follows:

$$f_{\delta,\pi}(\rho_t, \delta_t, \pi_t) \quad = \quad \rho_t \left(\frac{\pi_t}{\overline{\pi}} - 1\right)^2 + (1-\rho_t)\, D_t \left(\frac{\delta_t}{\overline{\delta}} - 1\right)^2 , \tag{21}$$

where $\overline{\pi}$ and $\overline{\delta}$ are the fixed target values and $\rho_t \in [0, 1]$ is a given, potentially time-dependent, weight parameter defining the intermediate system.

Proposition 1 (1^{st} level: optimal benefit ratio and contribution rate)**.** *The optimal benefit ratio $\delta_t^\star$ and optimal contribution rate $\pi_t^\star$ of the objective function* (21)*, for general $\overline{\delta}$ and $\bar{\pi}$, is given by:*

$$\delta_t^\star \quad = \quad \bar{\delta}\,\bar{\pi}\, \frac{\rho_t\, \overline{\delta} + (1-\rho_t)\overline{\pi}}{\rho_t\, D_t\, \overline{\delta}^2 + (1-\rho_t)\, \overline{\pi}^2}, \tag{22}$$

$$\pi_t^\star \quad = \quad D_t\, \delta_t^\star . \tag{23}$$

Proof. The optimal benefit ratio δ and the optimal contribution rate π are obtained by substitution of the PAYG equilibrium equation (eq. 6) in the objective function

$$f_{\delta,\pi}(\rho, \delta_t, D_t) = \rho_t \left(\frac{D_t\, \delta_t}{\overline{\pi}} - 1\right)^2 + (1-\rho_t)\, D_t \left(\frac{\delta_t}{\overline{\delta}} - 1\right)^2 .$$

[4] Other forms of objective function can be considered. For example, the exponential form $\exp\left[-\gamma\, \rho_t \left(\frac{\pi_t}{\overline{\pi}} - 1\right) - \gamma\,(1-\rho_t) \left(\frac{\delta_t}{\overline{\delta}} - 1\right)\right]$. In order to apply the proposed optimisation, the objective function has to be differentiable and convex. If it is convex but not differentiable, similar results can be numerically obtained.

Deriving with respect to δ_t, and cancelling the derivative yields:

$$\frac{\partial}{\partial\delta} f_{\delta,\pi}(\rho_t,\delta_t,D_t) = 2\,\rho_t \frac{D_t}{\overline{\pi}}\left(\frac{D_t\,\delta_t}{\overline{\pi}} - 1\right) + 2\,(1-\rho_t)\,D_t \frac{1}{\overline{\delta}}\left(\frac{\delta_t}{\overline{\delta}} - 1\right) = 0\ .$$

The optimal processes are deduced from this last relation and equation (6):

$$\begin{aligned} \delta_t^\star &= \bar{\delta}\,\bar{\pi}\,\frac{\rho_t\,\overline{\delta} + (1-\rho_t)\overline{\pi}}{\rho_t\,D_t\,\overline{\delta}^2 + (1-\rho_t)\,\overline{\pi}^2} && (24)\\ \pi_t^\star &= D_t\,\delta_t^\star\ . && (25) \end{aligned}$$

The second derivative of f is given by:

$$\frac{\partial^2}{\partial\delta^2} f_{\delta,\pi}(\rho_t,\delta_t,D_t) = 2\,D_t\left(\frac{\rho_t D_t}{\overline{\pi}^2} + \frac{1-\rho_t}{\bar{\delta}^2}\right) \tag{26}$$

which, for $\rho_t \in [0,1]$, is always positive indicating that the obtained $\delta_t^\star$ and corresponding $\pi_t^\star$ minimizes $f_{\delta,\pi}$. □

The optimal processes $\delta_t^\star$ and $\pi_t^\star$ depend on the dependency ratio D_t, the chosen risk-sharing weight ρ_t as well as the target values $\bar{\pi}$ and $\bar{\delta}$. In particular, $\delta_t^\star$ can be expressed as the target benefit ratio $\bar{\delta}$ multiplied by a time-dependent function

$$h(\rho_t,\bar{\pi},D_t) = \bar{\pi}\,\frac{\rho_t\,\overline{\delta} + (1-\rho_t)\overline{\pi}}{\rho_t\,D_t\,\overline{\delta}^2 + (1-\rho_t)\,\overline{\pi}^2}.$$

The function h is clearly decreasing with D_t indicating that, as the number of retirees with respect to the workers increase, the benefit ratio will have to adjust downwards accordingly. The function h depends on a convex transformation of the target benefit and contribution rate, albeit in different order as in the original objective function $f_{\delta,\pi}$ (21). Since $\bar{\delta}$, $\bar{\pi}$, ρ_t and $D_t \in [0,1]$, the denominator will always be substantially smaller than the numerator yielding $h(\rho_t,\bar{\pi},D_t) \geq \bar{\pi}\ \forall t$.

Corollary 1 (Optimal $\delta_t^\star$ and $\pi_t^\star$ when $\bar{D} = \frac{\bar{\pi}}{\bar{\delta}}$)**.** *Let $\bar{D} = \frac{\bar{\pi}}{\bar{\delta}}$ be the implicit dependency ratio that results from applying* (6) *to the targets $\bar{\pi}$ and $\bar{\delta}$. Then, the optimal benefit ratio $\delta_t^\star$* (22) *and contribution rate $\pi_t^\star$* (23) *can be rewritten as:*

$$\begin{aligned} \delta_t^\star &= \bar{\delta}\,\frac{\rho_t + (1-\rho_t)\bar{D}}{\rho_t\,\frac{D_t}{\bar{D}} + (1-\rho_t)\bar{D}}, && (27)\\ \pi_t^\star &= \bar{\pi}\,\frac{D_t}{\bar{D}}\,\frac{\rho_t + (1-\rho_t)\bar{D}}{\rho_t\,\frac{D_t}{\bar{D}} + (1-\rho_t)\bar{D}}\ . && (28) \end{aligned}$$

Proof. Obviously, taking $\overline{\delta}$ as common factor in the numerator and denominator of $\delta_t^\star$, and letting $\bar{D} = \frac{\bar{\pi}}{\bar{\delta}}$ we find

$$\begin{aligned} \delta_t^\star &= \bar{\delta}\,\bar{\pi}\,\bar{\delta}\,\frac{\rho_t\ + (1-\rho_t)\frac{\overline{\pi}}{\bar{\delta}}}{\overline{\delta}^2\left(\rho_t\,D_t\ + (1-\rho_t)\ \left(\frac{\overline{\pi}}{\bar{\delta}}\right)^2\right)} = \bar{\pi}\,\frac{\rho_t + (1-\rho_t)\bar{D}}{\rho_t\,D_t + (1-\rho_t)\bar{D}^2} = \frac{\bar{\pi}}{\bar{D}}\,\frac{\rho_t + (1-\rho_t)\bar{D}}{\rho_t\,\frac{D_t}{\bar{D}} + (1-\rho_t)\bar{D}}\\ &= \bar{\delta}\,\frac{\rho_t + (1-\rho_t)\bar{D}}{\rho_t\,\frac{D_t}{\bar{D}} + (1-\rho_t)\bar{D}} \end{aligned}$$

The expression for the optimal contribution rate follows from (23):

$$\pi_t^\star = \bar{\pi}\frac{D_t}{\bar{D}}\frac{\rho_t + (1-\rho_t)\bar{D}}{\rho_t \frac{D_t}{\bar{D}} + (1-\rho_t)\bar{D}}$$

□

Note that $\bar{D}$ does not necessarily coincide with an actual observed dependency ratio. It simply results of the PAYG equilibrium (6) applied to our target variables. Obviously, the simplified expression of Corollary 1 implies that

$$\begin{cases} \delta_t^\star < \bar{\delta} & \text{if} \quad D_t > \bar{D}, \\ \delta_t^\star > \bar{\delta} & \text{if} \quad D_t < \bar{D}. \end{cases} \tag{29}$$

Proposition 1 provides the optimal benefit ratio $\delta_t^\star$ and contribution rate $\pi_t^\star$ for general targets $\bar{\pi}$ and $\bar{\delta}$ and weights ρ_t. Our base case analysis that follows, which we denote *canonical* choice, is given by the particular choice of $\overline{\delta} = \delta_0$ and $\overline{\pi} = \pi_0$. Remark 3 shows that this *canonical* choice of targets yields naturally the pure DB and pure DC scenarios when ρ is either equal to 1 or 0, respectively. Equation (29) shows that, since $\bar{D} = D_0$, the optimal benefit ratio will decrease in an ageing environment since $D_t > D_0$. We further assess the effect of the targets and weight parameters in two examples. Example 1 below investigates the effect of the chosen targets for a given weight parameter $\rho = \frac{1}{2}$ corresponding to a democratic choice giving equal importance to workers and retirees' needs whereas Example 2 analyses the effect of varying $\rho \in [0,1]$.

Remark 3 (DB, DC vs intermediate). The extreme cases of pure DB and DC can be found by simplifying (22) and (23) through ρ_t. Indeed, if $\rho_t = 1$, that is, we are solely interested in studying the evolution of the contribution rate affecting the working age population, we obtain

$$\begin{aligned} \delta_t^\star &= \bar{\delta}\,\bar{\pi}\,\frac{\overline{\delta}}{D_t\,\overline{\delta}^2} = \frac{\bar{\pi}}{D_t}, \\ \pi_t^\star &= \bar{\pi}. \end{aligned}$$

If $\bar{\pi} = \pi_0$ this corresponds to a pure DC scheme where contribution rate remains unchanged and the benefit ratio is adjusted through (6) as shown in Equation (11) from Subsection 3.1. If, on the other hand, $\rho_t = 0$, we focus on the retirees benefit ratio and Proposition 1 yields

$$\begin{aligned} \delta_t^\star &= \bar{\delta}\,\bar{\pi}\,\frac{\overline{\pi}}{\overline{\pi}^2} = \bar{\delta}, \\ \pi_t^\star &= D_t\,\bar{\delta}. \end{aligned}$$

Then, we have a pure DB scheme if $\bar{\delta} = \delta_0$. The optimal contribution rate $\pi_t^\star$ simply corresponds to Equation (6). Finally, let us consider the case where equal weight is given to the π and δ stability, $\rho_t = \frac{1}{2}\,\forall t$. In this case, the optimal $\delta_t^\star$ and $\pi_t^\star$ become:

$$\begin{aligned} \delta_t^\star &= \bar{\delta}\frac{\frac{1}{2} + \frac{1}{2}\bar{D}}{\frac{1}{2}\frac{D_t}{\bar{D}} + \frac{1}{2})\bar{D}} = \bar{\delta}\frac{1+\bar{D}}{\frac{D_t}{\bar{D}} + \bar{D}}, \\ \pi_t^\star &= D_t\,\delta_t^\star = \bar{\pi}\frac{D_t}{\bar{D}}\frac{1+\bar{D}}{\frac{D_t}{\bar{D}} + \bar{D}}\ . \end{aligned}$$

Table 1: Target values for S1–S4 when $D_0 = 0.3$, $D_\infty = 0.5$, $\delta_0 = 0.5$ and $\pi_0 = 0.15$.

Scenario	S1	S2	S3	S4
$\bar{\delta}$	0.50	0.30	0.50	0.40
$\bar{\pi}$	0.15	0.15	0.25	0.20
$\bar{D}$	0.30	0.50	0.50	0.50

Of course, this further simplifies to

$$\delta_t^\star = \delta_0 \frac{1 + D_0}{\frac{D_t}{D_0} + D_0}, \tag{30}$$

$$\pi_t^\star = \pi_0 \frac{D_t}{D_0} \frac{1 + D_0}{\frac{D_t}{D_0} + D_0}\ . \tag{31}$$

for our *canonical* choice of targets. Further sensitivities to the choice of ρ are presented in Example 2. □

Example 1 (Evolution of $\delta_t^\star$ and $\pi_t^\star$ for scenarios S1–S4 (32)). To assess the effect of the choice of target, let us consider four possible cases (S1–S4) for our study:

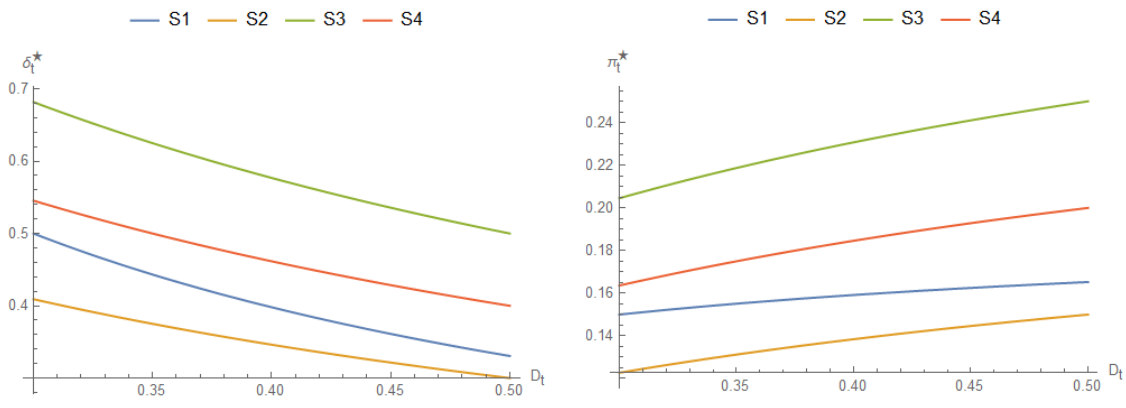
$$\begin{cases} \text{(S1)} & \bar{\delta} = \delta_0 & \text{and} & \bar{\pi} = \pi_0, \\ \text{(S2)} & \bar{\delta} = \frac{\pi_0}{D_\infty} & \text{and} & \bar{\pi} = \pi_0, \\ \text{(S3)} & \bar{\delta} = \delta_0 & \text{and} & \bar{\pi} = \bar{\delta}\, D_\infty, \\ \text{(S4)} & \bar{\delta} = \delta^* & \text{and} & \bar{\pi} = \pi^*, \end{cases} \tag{32}$$

where D_∞ is the long term dependency ratio. The first scenario, S1, aims at varying the contribution and benefit ratio without diverting too much from the initial PAYG equilibrium state at $t = 0$. This corresponds to our *canonical* choice. On the other hand, scenario S2 and S3 target either the initial contribution rate or initial benefit ratio respectively despite anticipated future costs under an ageing environment. The benefit ratio and contribution rate result from (6). On the other hand, S4 represents a scenario where the social planner would choose $\bar{\delta}$ and $\bar{\pi}$ not based on PAYG equilibrium arguments but on political choices. Note that D_∞ is equal to $\bar{D}$ in (S2) and (S3) by construction, but not necessarily equals $\bar{D}$ in general. For instance, in S1 we have $\bar{D} = D_0$ whereas D_∞ is the dependency ratio at the end of our studied horizon. These clearly do not coincide in an ageing environment. The weight parameter is given by $\rho = \frac{1}{2}$ for the four studied scenarios.

Let $D_0 = 0.3$ and $D_\infty = 0.5$, that is, we are in an ageing scenario. We choose $\delta_0 = 0.5$ without loss of generality.[5] This yields the $t = 0$ PAYG equilibrium initial parameters $\pi_0 = 0.15$ and $\delta_0 = 0.5$. S1 has hence $\bar{\pi} = \pi_0 = 0.15$ and $\bar{\delta} = \delta_0 = 0.5$, S2 has $\bar{\pi} = \pi_0 = 0.15$ and $\bar{\delta} = \frac{\pi_0}{D_\infty} = 0.3$ and S3 has $\bar{\delta} = \delta_0 = 0.5$ and $\bar{\pi} = \delta_0\, D_\infty = 0.25$. The planner's choice in S4 is $\bar{\delta} = 0.4$ and $\bar{\pi} = 0.2$ which corresponds to π_∞ and δ_∞ from Equations (14) and (13) when replacing $D_t = D_\infty$ under the Musgrave rule for a choice of $M = 0.5$. Anticipating ageing, the government chooses to target a lower than current benefit ratio and simultaneously a greater long-term contribution rate. The corresponding implied target dependency ratios are $\bar{D} = 0.3, 0.5, 0.5, 0.5$ for S1–S4 respectively. Table 1 provides an overview of the target values for S1–S4.

[5] We have assessed other values of δ_0 and obtain the same reasoning, albeit on a difference scale. This follows from calculating π_0 using the PAYG equilibrum (6) for a given δ_0. Of course, using this equivalence, greater δ_0 yield a greater π_0 for a given initial dependency ratio D_0.

Figure 1: Optimal benefit ratio $\delta_t^\star$ (22) and contribution rate $\pi_t^\star$ (23) for Scenarios S1–S4 (32)



(a) $\delta_t^\star$ (b) $\pi_t^\star$

Notes: S1–S4 represent scenarios given in Equation (32) for $\rho = 0.5$, $D_0 = 0.3$, $D_\infty = 0.5$, $\pi_0 = 0.15$ and $\delta_0 = 0.5$. Relevant parameters are given in Table 1.

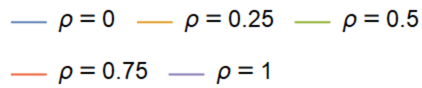
Figure 1 shows the great effect of the targets in the short and long-term evolution of $\delta_t^\star$ and $\pi_t^\star$. All scenarios yield to a decreasing benefit rate and increasing contribution rate in an ageing environment. However, as expected, the starting and final level vary substantially depending on the chosen targets. Let us start analysing S1 (blue line) which targets the initial equilibrium values of the PAYG system. In this case Figures 1(a) and 1(b) show that we indeed start at $\delta_t^\star = \delta_0 = 0.5$ and $\pi_t^\star = \pi_0 = 0.15$ as expected. Obviously, in this case $\bar{D} = D_0$ and at time $t = 0$ Equations (27) and (28) yield δ_0 and π_0 respectively.

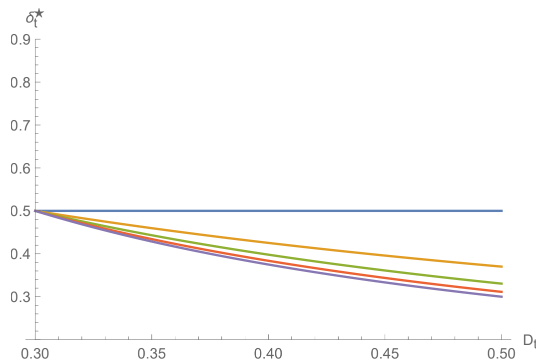
It is interesting to note that changing the targets dramatically alters the levels of benefits and contributions despite the initial $\delta_0 = 0.5$ and the corresponding π_0. For instance, in S2, we target 0.3 since we anticipate a significant increase in D_t and aim to adjust the benefit level accordingly without impacting contributions much. Of course, the pension system adapts to converge to 0.3 when $D_t = 0.5$. However, the initial payment level is no longer 0.5 but slightly above 0.4. Targeting lower long-term benefits adapts the system downwards, making it simultaneously less generous and relatively less expensive. This of course stems from the PAYG equilibrium (6): $\delta_0^\star$ being lower than 0.5, for $D_0 = 0.3$ we obtain necessarily $\pi_0^\star < \pi_0 = 0.15$. On the other hand, S3's philosophy is different. Anticipating an ageing society, it aims to adapt contributions as to guarantee a level similar to $\delta_0 = 0.5$ in the long-term. By doing so, it increases the initial benefit ratio to slightly under 0.7 and increases the contribution rate to 20% instead of 15% to maintain PAYG equilibrium and optimality. Finally, S4 shows that, affecting contributors and retirees simultaneously in the long-term through the chosen targets, yields reasonable benefits and contributions in the short-term as well. Figure 1 shows clearly that $\delta_t^\star > \bar{\delta}$ for S2–S4 since $\bar{D} \geq D_t \, \forall t$ whereas $\delta_t^\star < \bar{\delta}$ since $\bar{D} = D_0 \leq D_t$ as discussed in Equation (29). □

Example 2 (Benefit ratio $\delta_t^\star$ and contribution rate $\pi_t^\star$ as a function of the weight parameter ρ for scenarios S1 and S4 (32)). Figure 2 shows the optimal benefit ratio $\delta_t^\star$ (22) and contribution rate $\pi_t^\star$ (23) for varying ρ and scenarios S1 and S4.[6] Firstly, we observe that the benefit ratio decreases and the contribution rate increases under an ageing scenario where D_t goes from

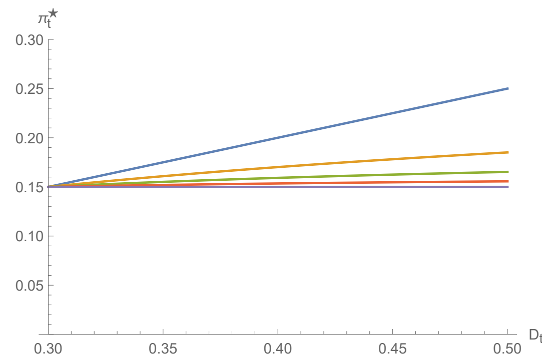
[6]Scenarios 2 and 3 are omitted from the main analysis as they resemble Scenario 4, albeit with a different scale given the different targets.

Figure 2: Optimal benefit ratio $\delta_t^\star$ (22) and contribution rate $\pi_t^\star$ (23) for varying ρ and scenarios S1 and S4

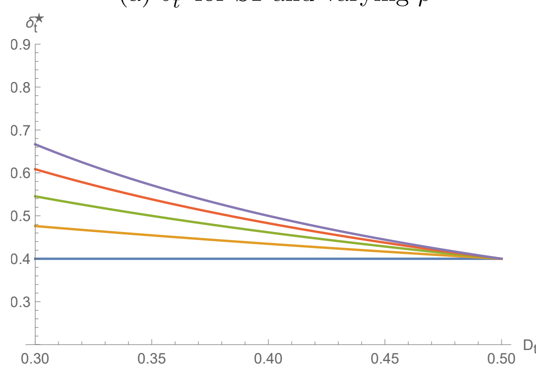




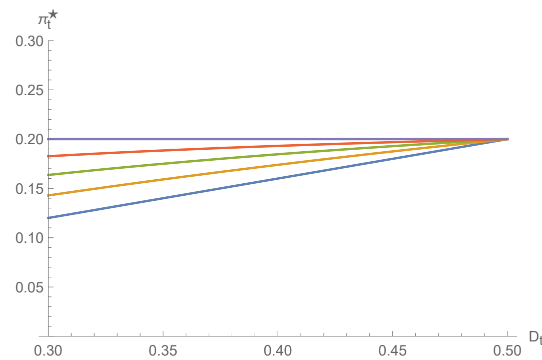
(a) $\delta_t^\star$ for S1 and varying ρ



(b) $\pi_t^\star$ for S1 and varying ρ



(c) $\delta_t^\star$ for S4 and varying ρ



(d) $\pi_t^\star$ for S4 and varying ρ

Notes: S1–S4 parameters are given in Table 1. The parameter weight parameter $\rho \in [0, 1]$.

$D_0 = 0.3$ to $D_\infty = 0.5$. The level and speed of change heavily depends on the chosen targets as discussed in detail in Example 1.

We know that $\delta_t^\star = \bar{\delta}$ when $\rho = 0$ and $\pi_t^\star = \bar{\pi}$ when $\rho = 1$ from Remark 3. In particular, in S1, we recover the pure DB case ($\rho = 0$) and pure DC case ($\rho = 1$) since $\bar{\delta} = \delta_0$ and $\bar{\pi} = \pi_0$. The level of ρ indicates the level and speed at which we move from a pure DB to a pure DC. Since $\bar{D} = D_0 < D_t \, \forall \, t > 0$ we find that $\delta_t^\star < \delta_0$ as given in (29). The benefit ratio decreases up to 20% and the contribution rate increases up to 10% over the studied horizon.

On the other hand, the implicit dependency ratio $\bar{D}$ is equal to D_∞ in S4, yielding for all ρ $\delta_t^\star > \bar{\delta}$. The extreme cases yield here also a DB and DC scenario but for different initial values of the optimal benefit and contribution rate. Indeed, in the "pessimistic" DC, we target 20% and for $\rho = 1$ this is also the level of the initial contribution rate. Of course, since $D_0 = 0.3$, the corresponding benefit rate that renders the PAYG system sustainable exceeds $\delta_0 = 0.5$. Hence, in this pure DC we start higher than currently aimed but in the long term decrease the global benefit payments by 30%. However, the level of benefits is still higher than in S1 because the target contribution rate in this DC is 5% greater than the one aimed in S1. In summary, the level of $\bar{D}$, and how it compares to D_t is the main driver of the level of decrease from the chosen targets. □

4.2 Second level: risk sharing between retirees

This Subsection focuses on finding the optimal sustainability factor β and replacement rate at retirement age $\tilde{\delta}$ such that the effect of PAYG equilibrium imbalances (6) are shared between new and old retirees. We assume that the sustainability factor β can be rewritten as a product of past cumulated sustainability factors by the one-year sustainability factor $\tilde{\beta}_t = \beta^1_{t-1}$ that affects all old retirees at time t:

$$\beta^{\Delta}_t = \beta^{\Delta-1}_t \tilde{\beta}_{t+\Delta} = \prod_{s=t}^{t+\Delta-1} \beta^1_s = \prod_{s=t+1}^{t+\Delta} \tilde{\beta}_s, \tag{33}$$

where $\tilde{\beta}_s$ represents the sustainability factor applied to year s. Then, the replacement rate $\tilde{\delta}_t$ for the cohort retiring in t can be found by rewriting the benefit ratio (7):

$$\begin{aligned}
\delta_t &= \tilde{\delta}_t\, l_{x_r,t}\, \beta^0_t + \sum_{x=x_r+1}^{\omega} \widetilde{\delta}_{t-(x-x_r)}\, \beta^{x-x_r-1}_{t-(x-x_r)}\, \tilde{\beta}_t\, l_{x,t} \Leftrightarrow \\
\tilde{\delta}_t &= \frac{\delta_t - \sum_{x=x_r+1}^{\omega} \widetilde{\delta}_{t-(x-x_r)}\, \beta^{x-x_r-1}_{t-(x-x_r)}\, \tilde{\beta}_t\, l_{x,t}}{l_{x_r,t}} = \frac{\delta_t - \tilde{\beta}_t\, \alpha_t}{l_{x_r,t}},
\end{aligned} \tag{34}$$

where α_t represents the sum of pensions to old retirees before year t $\tilde{\beta}_t$ adjustment and is given by

$$\alpha_t = \sum_{x=x_r+1}^{\omega} \widetilde{\delta}_{t-(x-x_r)}\, \beta^{x-x_r-1}_{t-(x-x_r)}\, l_{x,t}. \tag{35}$$

The replacement rate $\tilde{\delta}_t$ (34) is a function of the benefit ratio δ_t obtained in the first level, the level of pensions for old retirees, the sustainability factor $\tilde{\beta}_t$ corresponding to year t and the density corresponding to new retirees $l_{x_r,t}$. Alternatively, rewriting the benefit ratio (7) to highlight the one-year sustainability factor $\tilde{\beta}_t$ yields

$$\tilde{\beta}_t = \frac{\delta_t - \tilde{\delta}_t\, l_{x_r,t}}{\alpha_t}. \tag{36}$$

Following the same approach as the first level, we want to minimize the following objective function

$$g(\eta_t, \widetilde{\delta}_t, \widetilde{\beta}_t) = \eta_t \sum_{x_r+1}^{\omega} L_{x,t} \left(\frac{\widetilde{\beta}_t}{\overline{\overline{\beta}}_t} - 1 \right)^2 + (1-\eta_t)\, L_{x_r,t} \left(\frac{\widetilde{\delta}_t}{\overline{\overline{\delta}}_t} - 1 \right)^2 \tag{37}$$

Since $\sum_{x_r+1}^{\omega} L_{x,t} > 0$, the optimum obtained through (37) is equivalent to the optimum obtained through $\frac{g(\eta_t, \widetilde{\delta}_t, \widetilde{\beta}_t)}{\sum_{x_r+1}^{\omega} L_{x,t}}$. Let us denote

$$D_{x_r,t} = \frac{L_{x_r,t}}{\sum_{x_r+1}^{\omega} L_{x,t}}, \tag{38}$$

as the dependency ratio of newly retirees towards old retirees. Recall that $L_{x,t}$ denotes the number of individuals whereas $l_{x,t}$ represents the density. Then, (37) can be rewritten as follows:

$$g_{\tilde{\delta},\tilde{\beta}}(\eta_t, \widetilde{\delta}_t, \widetilde{\beta}_t) = \eta_t \left(\frac{\widetilde{\beta}_t}{\overline{\overline{\beta}}_t} - 1 \right)^2 + (1-\eta_t)\, D_{x_r,t} \left(\frac{\widetilde{\delta}_t}{\overline{\overline{\delta}}_t} - 1 \right)^2. \tag{39}$$

Proposition 2 (2^{nd} level: optimal replacement rate $\tilde{\delta}$ and sustainability factor $\tilde{\beta}$ for target $\overline{\beta}_t$ and $\overline{\overline{\delta}}_t$)**.** *The optimal replacement rate $\tilde{\delta}_t^\star$ and sustainability factor $\tilde{\beta}_t^\star$ of the objective function* (39) *is given by:*

$$\tilde{\beta}_t^\star = \overline{\beta}_t \frac{\eta_t \overline{\overline{\delta}}_t^2 \, l_{x_r,t}^2 + (1-\eta_t)\,\alpha_t \,\overline{\beta}_t \, D_{x_r,t}(\delta_t - \overline{\overline{\delta}}_t \, l_{x_r,t})}{\eta_t \overline{\overline{\delta}}_t^2 \, l_{x_r,t}^2 + (1-\eta_t)\,\alpha_t^2 \,\overline{\beta}_t^2 \, D_{x_r,t}}, \tag{40}$$

$$\tilde{\delta}_t^\star = \overline{\overline{\delta}}_t \frac{\eta_t \overline{\overline{\delta}}_t \, l_{x_r,t}(\delta_t - \alpha_t \overline{\beta}_t) + (1-\eta_t)\,\alpha_t^2 \overline{\beta}_t^2 \, D_{x_r,t}}{\eta_t \overline{\overline{\delta}}_t^2 \, l_{x_r,t}^2 + (1-\eta_t)\,\alpha_t^2 \,\overline{\beta}_t^2 \, D_{x_r,t}}. \tag{41}$$

Proof. The optimal sustainability factor $\tilde{\beta}$ and the optimal replacement rate $\tilde{\delta}$ are obtained by substituting $\widetilde{\delta}_t$ by (34) in the objective function

$$g_{\tilde{\delta},\tilde{\beta}}(\eta, \widetilde{\delta}_t, \widetilde{\beta}_t) = \eta_t \left(\frac{\widetilde{\beta}_t}{\overline{\beta}_t} - 1 \right)^2 + (1-\eta_t)\, D_{x_r,t} \left(\frac{\delta_t - \tilde{\beta}_t \alpha_t}{l_{x_r,t} \overline{\overline{\delta}}_t} - 1 \right)^2$$

Deriving with respect to $\tilde{\beta}_t$, and cancelling the derivative yields:

$$\frac{\partial}{\partial \tilde{\beta}_t} g_{\tilde{\delta},\tilde{\beta}}(\eta, \widetilde{\delta}_t, \widetilde{\beta}_t) = 2\,\eta_t \frac{\tilde{\beta}_t - \overline{\beta}_t}{\overline{\beta}_t^2} + 2\,(1-\eta_t)\,\alpha_t D_{x_r,t} \left(\frac{\alpha_t \tilde{\beta}_t - \delta_t + \overline{\overline{\delta}}_t l_{x_r,t}}{\overline{\overline{\delta}}_t^2 l_{x_r,t}^2} \right) = 0.$$

The optimal processes are deduced from this last relation and equation (34):

$$\tilde{\beta}_t^\star = \overline{\beta}_t \frac{\eta_t \overline{\overline{\delta}}_t^2 \, l_{x_r,t}^2 + (1-\eta_t)\,\alpha_t \,\overline{\beta}_t \, D_{x_r,t}(\delta_t - \overline{\overline{\delta}}_t \, l_{x_r,t})}{\eta_t \overline{\overline{\delta}}_t^2 \, l_{x_r,t}^2 + (1-\eta_t)\,\alpha_t^2 \,\overline{\beta}_t^2 \, D_{x_r,t}}$$

$$\tilde{\delta}_t^\star = \overline{\overline{\delta}}_t \frac{\eta_t \overline{\overline{\delta}}_t \, l_{x_r,t}(\delta_t - \alpha_t \overline{\beta}_t) + (1-\eta_t)\,\alpha_t^2 \overline{\beta}_t^2 \, D_{x_r,t}}{\eta_t \overline{\overline{\delta}}_t^2 \, l_{x_r,t}^2 + (1-\eta_t)\,\alpha_t^2 \,\overline{\beta}_t^2 \, D_{x_r,t}}$$

The second derivative of f is given by:

$$\frac{\partial^2}{\partial \tilde{\beta}_t^2} g_{\tilde{\delta},\tilde{\beta}}(\eta_t, \widetilde{\delta}_t, \widetilde{\beta}_t) = 2 \left(\frac{\eta_t}{\overline{\beta}_t^2} + \frac{(1-\eta_t)\alpha_t^2 D_{x_r,t}}{\overline{\overline{\delta}}_t^2 l_{x_r,t}^2} \right) \tag{42}$$

which, for $\eta_t \in [0,1]$, is always positive indicating that the obtained $\tilde{\beta}_t^\star$ and corresponding $\tilde{\delta}_t^\star$ minimizes $g_{\tilde{\delta},\tilde{\beta}}$. □

As in Proposition 1, the choice of the targets is important. Our *canonical* choice in the 2nd level corresponds to the following two objectives: letting the target replacement rate be equal to the benefit ratio stemming from the 1st level, and the target sustainability factor to be equal to 1 as follows:

$$\begin{cases} \overline{\overline{\delta}}_t &= \delta_t, \\ \overline{\beta}_t &= 1. \end{cases} \tag{43}$$

This choice corresponds to the cases studied in Subsection 3.2. Under this choice, our optimal $\tilde{\beta}_t$ and $\tilde{\delta}_t$ simplify as follows:

$$\tilde{\beta}_t^\star = \delta_t \frac{\eta_t\, \delta_t\, l_{x_r,t}^2 + (1-\eta_t)\, \alpha_t\, D_{x_r,t}\, (1-l_{x_r,t})}{\eta_t\, \delta_t^2\, l_{x_r,t}^2 + (1-\eta_t)\, \alpha_t^2\, D_{x_r,t}}, \tag{44}$$

$$\tilde{\delta}_t^\star = \delta_t \frac{\eta_t\, \delta_t\, l_{x_r,t}(\delta_t - \alpha_t) + (1-\eta_t)\, \alpha_t^2\, D_{x_r,t}}{\eta_t\, \delta_t^2\, l_{x_r,t}^2 + (1-\eta_t)\, \alpha_t^2\, D_{x_r,t}}. \tag{45}$$

Remark 4 (Simplifying $\tilde{\beta}_t^\star$ (44) under the *canonical* choice). Let us rewrite α_t, the sum of pensions to older retirees before adjusting with $\tilde{\beta}_t$ to simplify (44) and (45) further:

$$\begin{aligned}\alpha_t &= \sum_{x=x_r+1}^{\omega} \tilde{\delta}_{t-(x-x_r)}\, \beta_{t-(x-x_r)}^{x-x_r-1}\, l_{x,t} = \frac{l_{x_r,t}}{l_{x_r,t}} \sum_{x=x_r+1}^{\omega} \tilde{\delta}_{t-(x-x_r)}\, \beta_{t-(x-x_r)}^{x-x_r-1}\, l_{x,t} \\ &= l_{x_r,t} \sum_{x=x_r+1}^{\omega} \tilde{\delta}_{t-(x-x_r)}\, \beta_{t-(x-x_r)}^{x-x_r-1}\, \frac{l_{x,t}}{l_{x_r,t}} = l_{x_r,t}\alpha_t^*, \end{aligned} \tag{46}$$

where $\frac{l_{x,t}}{l_{x_r,t}}$ is only equal to a survival probability[7] if there is no time-dependence. Note that α_t^* resembles an indexed lifelong annuity factor where indexation corresponds to the β cumulative sustainability factor and $\tilde{\delta}$ to the level of benefits. The dependency ratio $D_{x_r,t}$ can also be written with regards to $l_{x_r,t}$ as follows:

$$D_{x_r,t} = \frac{L_{x_r,t}}{\sum_{x_r+1}^{\omega} L_{x,t}} = \frac{\sum_{x=x_r}^{\omega} l_{x,t}}{\sum_{x=x_r}^{\omega} l_{x,t}} \frac{L_{x_r,t}}{\sum_{x_r+1}^{\omega} L_{x,t}} = \frac{l_{x_r,t}}{\sum_{x_r+1}^{\omega} l_{x,t}}. \tag{47}$$

Furthermore, note that $\sum_{x=x_r}^{\omega} l_{x,t} = 1$ and hence $1 - l_{x_r,t} = \sum_{x_r+1}^{\omega} l_{x,t}$. Considering this, we obtain the simplified optimal sustainability factor:

$$\tilde{\beta}_t^\star = \delta_t \frac{\eta_t\, \delta_t\, + (1-\eta_t)\, \alpha_t^*}{\eta_t\, \delta_t^2 + (1-\eta_t)\, (\alpha_t^*)^2\, D_{x_r,t}}. \tag{48}$$

The sustainability factor corresponds to the benefit level of the 1st level adjusted by a convex transformation of the benefit ratio one should provide on an aggregate base and the total ex-ante sustainability benefits for old retirees relative to new retirees. □

Remark 5 (Limit cases when $\eta_t = 0$ and $\eta_t = 1$). We are interested in seeing what our risk-sharing mechanism entails whenever η_t takes the extreme values 0 and 1. When $\eta_t = 0$, that is, we are solely interested in studying the evolution of the replacement rate, we obtain

$$\begin{aligned}\tilde{\beta}_t^\star &= \frac{\delta_t - \bar{\bar{\delta}}_t\, l_{x_r,t}}{\alpha_t}, \\ \tilde{\delta}_t^\star &= \bar{\bar{\delta}}_t\ ,\end{aligned}$$

whereby $\tilde{\beta}_t^\star$ is given by (36) with $\tilde{\delta}_t = \bar{\bar{\delta}}_t$. In particular, for the *canonical* choice (43), we obtain

$$\begin{aligned}\tilde{\beta}_t^\star &= \frac{\delta_t\, (1-l_{x_r,t})}{\alpha_t} = \frac{\delta_t\, (1-l_{x_r,t})}{\alpha_t^*\, l_{x_r,t}} = \frac{\delta_t}{\alpha_t^*} \frac{\sum_{x=x_r+1}^{\omega} l_{x,t}}{l_{x_r,t}}, \\ \tilde{\delta}_t^\star &= \delta_t.\end{aligned}$$

[7] The survival probability from x_r in t to x in $t - x_r + x$ would be ${}_{x-x_r}p_{x_r}(t) = \frac{l_{x,t+(x-x_r)}}{l_{x_r,t}}$.

This scenario is comparable to providing a replacement rate that is sustainable on a PAYG equilibrium basis to new retirees and obtain the sustainability factor that is able to finance this. Indeed, the *canonical* choice makes obvious that the sustainability rate will be equal to the relationship between the payment capacity from the 1st level (benefit ratio times current retirees) and the actual level of pensions to current retirees before any adjustment. This corresponds to Subsection 3.2 where solidarity is assumed within all retirees (Equation (18)).

On the other hand, if $\eta_t = 1$, that is, we are solely interested in studying the sustainability factor, we obtain

$$\begin{aligned}\tilde{\beta}_t^\star &= \overline{\beta}_t,\\ \tilde{\delta}_t^\star &= \frac{\delta_t - \alpha_t \overline{\beta}_t}{l_{x_r,t}}\ .\end{aligned}$$

As in the previous case, $\tilde{\delta}_t^\star = \tilde{\delta}_t$ given by Equation (34) for $\tilde{\beta}_t = \overline{\beta}_t$. For the *canonical* choice (43), yields exactly

$$\begin{aligned}\tilde{\beta}_t^\star &= 1,\\ \tilde{\delta}_t^\star &= \frac{\delta_t - \alpha_t}{l_{x_r,t}}.\end{aligned}$$

This scenario corresponds to the newly retirees bearing all the risk as presented in (17) in Subsection 3.2. Of course, as discussed then, we see that the replacement rate will be equal to the remaining payment capacity δ_t after providing full indexation to current retirees proportional to new retirees. Our framework presents possible intermediate cases between the extreme solidarity scenarios presented in Subsection 3.2. □

Remark 6 (Democratic choice when $\eta = \frac{1}{2}$). We now focus on the sustainability factor $\tilde{\beta}_t^\star$ and replacement rate $\tilde{\delta}_t^\star$ when equal weight is given to the two objectives, i.e., $\eta = \frac{1}{2}$. In this case, Equations (40) and (41), for general targets, become:

$$\tilde{\beta}_t^\star = \overline{\beta}_t \frac{\overline{\overline{\delta}}_t^2\, l_{x_r,t}^2 + \alpha_t\, \overline{\beta}_t\, D_{x_r,t}(\delta_t - \overline{\overline{\delta}}_t\, l_{x_r,t})}{\overline{\overline{\delta}}_t^2\, l_{x_r,t}^2 + \alpha_t^2\, \overline{\beta}_t^2\, D_{x_r,t}}, \tag{49}$$

$$\tilde{\delta}_t^\star = \overline{\overline{\delta}}_t \frac{\overline{\overline{\delta}}_t\, l_{x_r,t}(\delta_t - \alpha_t \overline{\beta}_t) + \alpha_t^2 \overline{\beta}_t^2\, D_{x_r,t}}{\overline{\overline{\delta}}_t^2\, l_{x_r,t}^2 + \alpha_t^2\, \overline{\beta}_t^2\, D_{x_r,t}}. \tag{50}$$

In particular, for our *canonical* choice, using (35) and (47), we obtain:

$$\tilde{\beta}_t^\star = \frac{\delta_t^2\, l_{x_r,t}^2 + \alpha_t\, D_{x_r,t}\, \delta_t\, (1 - l_{x_r,t})}{\delta_t^2\, l_{x_r,t}^2 + \alpha_t^2\, D_{x_r,t}} = \frac{\delta_t^2\, l_{x_r,t}^2 + \alpha_t\, \delta_t\, l_{x_r,t}}{\delta_t^2\, l_{x_r,t}^2 + \alpha_t^2\, \frac{l_{x_r,t}}{(1-l_{x_r,t})}} = \frac{\delta_t^2\, l_{x_r,t}}{\delta_t^2\, l_{x_r,t}} \frac{l_{x_r,t} + \frac{\alpha_t}{\delta_t}}{l_{x_r,t} + \left(\frac{\alpha_t}{\delta_t}\right)^2 \frac{1}{(1-l_{x_r,t})}}$$

$$= \frac{l_{x_r,t} + \frac{\alpha_t}{\delta_t}}{l_{x_r,t} + \left(\frac{\alpha_t}{\delta_t}\right)^2 \frac{1}{(1-l_{x_r,t})}} = \frac{1 + \frac{\alpha_t^*}{\delta_t}}{1 + \left(\frac{\alpha_t^*}{\delta_t}\right)^2 D_{x_r,t}} \tag{51}$$

$$\tilde{\delta}_t^\star = \delta_t \frac{\delta_t\, l_{x_r,t}(\delta_t - \alpha_t) + \alpha_t^2\, D_{x_r,t}}{\delta_t^2\, l_{x_r,t}^2 + \alpha_t^2\, D_{x_r,t}} = \delta_t \frac{\delta_t\, l_{x_r,t}}{\delta_t^2\, l_{x_r,t}} \frac{\delta_t - \alpha_t + \left(\frac{\alpha_t^2}{\delta_t}\right) \frac{D_{x_r,t}}{l_{x_r,t}}}{l_{x_r,t} + \left(\frac{\alpha_t}{\delta_t}\right)^2 \frac{D_{x_r,t}}{l_{x_r,t}}}$$

$$= \delta_t \frac{1 - \frac{\alpha_t}{\delta_t} + \left(\frac{\alpha_t}{\delta_t}\right)^2 \frac{1}{1-l_{x_r,t}}}{l_{x_r,t} + \left(\frac{\alpha_t}{\delta_t}\right)^2 \frac{1}{1-l_{x_r,t}}} = \delta_t \frac{\frac{1}{l_{x_r,t}} - \frac{\alpha_t^*}{\delta_t} + \left(\frac{\alpha_t^*}{\delta_t}\right)^2 D_{x_r,t}}{1 + \left(\frac{\alpha_t^*}{\delta_t}\right)^2 D_{x_r,t}}. \tag{52}$$

We observe that, $\tilde{\delta}_t^\star \geq \delta_t$ if and only if:

$$1 - \frac{\alpha_t}{\delta_t} \geq l_{x_r,t} \Leftrightarrow \delta_t - \alpha_t \geq l_{x_r,t}\, \delta_t, \tag{53}$$

that is, if the benefit ratio reduced by the pre-adjustment old retirees benefits is greater or equal than the benefit ratio weighted by the new retirees density.

Example 3 (Optimal replacement rate $\tilde{\delta}_t^\star$ (41) and sustainability factor (40) $\tilde{\beta}_t^\star$ for varying η). Let $\omega = 130$, $L_{x_r,t} = 1$ and $L_{x,t} = 0.96^{x-x_r}$. This yields $\sum_{x=x_r}^{\omega} L_{x,t} = 23.31$. Since the population densities are given by (8), then $l_{x_r,t} = 4.29 \cdot 10^{-2}$ and $D_{x_r,t} = 4.49 \cdot 10^{-2}$. In this case, if $\tilde{\delta}_{t-(x-x_r)} = \tilde{\delta} = 0.5$ and $\beta_{t-(x-x_r)}^{x-x_r} = 1$ for all old retirees, α_t becomes:

$$\alpha_t = \sum_{x=x_r+1}^{\omega} \tilde{\delta}_{t-(x-x_r)}\, \beta_{t-(x-x_r)}^{x-x_r-1}\, l_{x,t} = \tilde{\delta} \sum_{x=x_r+1}^{\omega} l_{x,t} = \tilde{\delta}(1 - l_{x_r,t})$$
$$= 0.5 \cdot 0.9571 = 0.4786$$

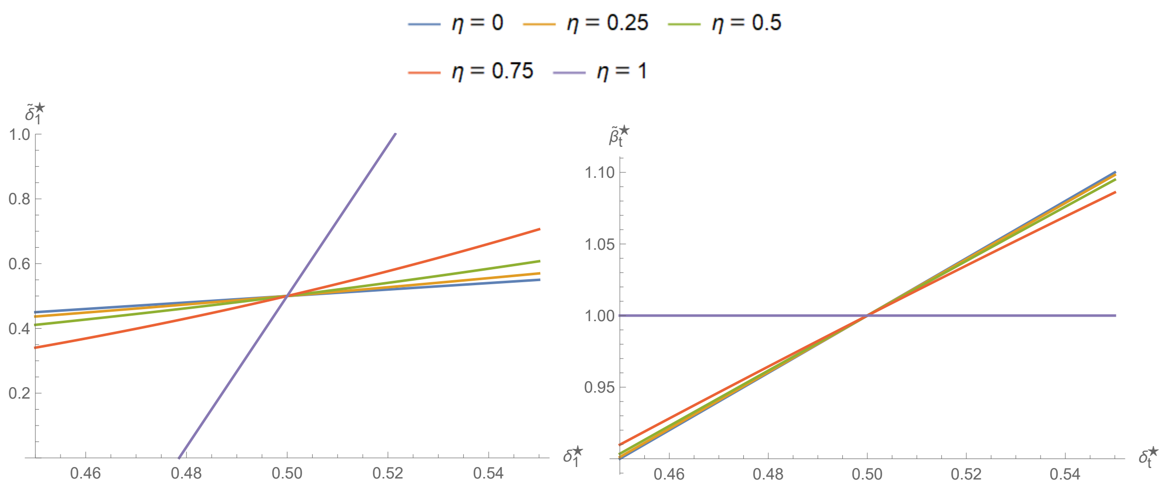
We work in a scenario where the past was virtually steady state, providing stable $\tilde{\delta} = 0.5$ and full wage indexation $\tilde{\beta} = 1$. Assume that, in year t, the 1st level analysis yields an average benefit ratio $\delta_t \in [0.45, 0.55]$. Recall that the *canonical* $\delta_t^\star$ is given by (30):

$$\delta_t^\star = \delta_0 \frac{1 + D_0}{\frac{D_t}{D_0} + D_0}.$$

Note that, since $\delta_0 = 0.5$ and $D_0 = 0.3$, the only way to reach after one year the lower bound $\delta_1^\star = 0.45$ is to have a 14.4% increase of the dependency ratio to $D_1 = 0.34$. This of course implies that, for a constant level of workers, retirees increase by 14.4% as well. Such a one-year change in the dependency would be not realistic, even in a rapidly ageing context.[8] First, we observe that for $\delta_t = 0.5$ we maintain $\tilde{\delta}_t = 0.5$ and $\tilde{\beta}_t = 1\, \forall \eta$. Indeed, this is where the curves cross.

[8] For instance, in our data, we observe an increase of the dependency ratio from 30% to 34% over a 7-year horizon.

Figure 3: Optimal replacement rate $\tilde{\delta}_t^\star$ (40) and sustainability factor $\tilde{\beta}_t^\star$ (40) for $\eta \in [0,1]$ and $\delta_t \in [0.45, 0.55]$



(a) $\tilde{\delta}_t^\star$ for $\eta \in [0,1]$

(b) $\tilde{\beta}_t^\star$ for $\eta \in [0,1]$

Notes: We assume $\omega = 130$, $L_{x_r,t} = 1$ and $L_{x,t} = 0.96^{x-x_r}$. The needed parameters then correspond to $l_{x_r,t} = 4.29 \cdot 10^{-2}$ and $D_{x_r,t} = 4.49 \cdot 10^{-2}$. Assuming that old retirees all had $\tilde{\beta}_{t-(x-x_r)} = \tilde{\delta} = 0.5$ and $\beta_{t-(x-x_r)}^{x-x_r} = 1$, α_t becomes 0.4786.

Let us assess the extreme cases $\eta = 0$ and $\eta = 1$ discussed in Remark 5. If $\eta = 1$ our optimal sustainability factor equals the target of full pension indexation. Of course, in counterpart, the replacement rate for new retirees is greatly affected by small changes in δ_t. In particular, the pension for new retirees becomes negative if $\delta_1 < 0.48$ which corresponds to a realistic one-year increase in retirees of 5.42%. It is clear that solely targeting stable indexation is not a feasible extreme scenario. On the other hand, when $\eta = 0$, we target a replacement rate equal to δ_t and the sustainability factor results from the equilibrium equation (7). In this case, a value of $\delta_1 = 0.48$ yields a replacement rate of $\tilde{\delta}_1 = 0.48$ and sustainability factor of $\tilde{\beta}_1 = 0.97$. Both parties see their benefits decrease but in a reasonable manner given the demographic change.

If $\delta_t < \tilde{\delta}$, then necessarily the replacement rate needs to decrease and full indexation cannot be guaranteed anymore. Interestingly, $\tilde{\delta}_t^\star$ is more affected by the choice of η than the sustainability factor β. Of course, demographic change yields additional ($\delta_t > 0.5$) or diminished indexation if $\delta_t < 0.5$. Indeed, abstracting from $\eta = 0$ and $\eta = 1$, we observe that the $\tilde{\beta}$ lines are much closer to each other than the corresponding $\tilde{\delta}$. Note that our 2nd level optimization compares the needs of one generation that represents around 4.5% of the total retirees versus the remaining 95.5%. In order words, it is intuitive that the variable that affects only one generation ($\tilde{\delta}$) is more sensitive to 1st level changes than $\tilde{\beta}$ which virtually affects all current retirees. Indeed, the marginal effect of ± 0.1 in the benefit ratio will be much greater for the newly retirees as they absorb in a great proportion the existing surplus or deficit. In other words, a small change in the sustainability factor $\tilde{\beta}$ can render an adequate initial replacement rate possible. Therefore we can conculde that the introduction of a sustainability factor adjusting the indexation of pension is a valuable automatic adjustment mechanism. $\square$

4.3 Initializing the risk-sharing mechanism

The expressions in Subsection 4.1 and 4.2 have been obtained for general pension and demographic structures. This concluding Subsection focuses on the first time application of the risk-sharing mechanism when the pension system goes from steady state to a dynamic scenario under the *canonical* choice with $\rho = \eta = \frac{1}{2}$. Our aim is to understand the main risk-sharing mechanism under simplified expressions.

Let us assume that the system was initially in steady state, that is,

$$D_s = \frac{R_s}{W_s} = D,$$
$$D_{x_r,s} = \frac{l_{x_r,s}}{\sum_{x=x_r+1}^{\omega} l_{x,s}} = D_{x_r}.$$

The system facing no demographic risk provides both stable indexation $\tilde{\beta}_s = 1$ and replacement rate $\tilde{\delta}_s = \tilde{\delta}$ for $s < t$. Of course, Equation (7) then simplifies to

$$\delta_s = \tilde{\delta} \sum_{x=x_r}^{\omega} l_{x,s} = \tilde{\delta} = \delta_0. \tag{54}$$

In that case, Equation (6) yields $\pi_s = D\,\delta = \pi$. The system runs in relative steady state for some years until t, where $D_t > D_{t-1} = D_{t-2} = \ldots = D_0 = D$. On the first level, the demographic change will yield, rewriting Equations (22) and (23):

$$\delta_t^\star = \tilde{\delta} \frac{1+D}{\frac{D_t}{D} + D}, \tag{55}$$
$$\pi_t^\star = \pi \frac{D_t}{D} \frac{1+D}{\frac{D_t}{D} + D}. \tag{56}$$

Of course, in our ageing scenario we have $\frac{D_t}{D} > 1$ and hence $\delta_t^\star < \tilde{\delta}$ and $\pi_t^\star > \pi$. It is obvious that if $D_t = D$ the optimal 1st level would simplify to $\delta_t^\star = \tilde{\delta}$ and $\pi_t^\star = \pi$. The variation in the 1st level needs to be translated in adjusted $\tilde{\delta}_t$ and $\tilde{\beta}_t$. First, let us simplify α_t (35):

$$\alpha_t = \sum_{x=x_r+1}^{\omega} \widetilde{\delta}_{t-(x-x_r)}\, \beta_{t-(x-x_r)}^{x-x_r-1}\, l_{x,t} = \widetilde{\delta} \sum_{x=x_r+1}^{\omega} l_{x,t} = \widetilde{\delta}\,(1 - l_{x_r,t}). \tag{57}$$

Then, Equation (51) simplifies to:

$$\begin{aligned}
\tilde{\beta}_t^\star &= \frac{l_{x_r,t} + \frac{\alpha_t}{\delta_t^\star}}{l_{x_r,t} + \left(\frac{\alpha_t}{\delta_t^\star}\right)^2 \frac{1}{(1-l_{x_r,t})}} = \frac{l_{x_r,t} + \frac{\tilde{\delta}}{\delta_t^\star}(1-l_{x_r,t})}{l_{x_r,t} + \left(\frac{\tilde{\delta}(1-l_{x_r,t})}{\delta_t^\star}\right)^2 \frac{1}{(1-l_{x_r,t})}} \\
&= \frac{l_{x_r,t} + \frac{\tilde{\delta}}{\delta_t^\star}(1-l_{x_r,t})}{l_{x_r,t} + \left(\frac{\tilde{\delta}}{\delta_t^\star}\right)^2 (1-l_{x_r,t})} = \frac{(1-l_{x_r,t})}{(1-l_{x_r,t})} \frac{D_{x_r,t} + \frac{\tilde{\delta}}{\delta_t^\star}}{D_{x_r,t} + \left(\frac{\tilde{\delta}}{\delta_t^\star}\right)^2} = \frac{D_{x_r,t} + \frac{\tilde{\delta}}{\delta_t^\star}}{D_{x_r,t} + \left(\frac{\tilde{\delta}}{\delta_t^\star}\right)^2} \\
&= \frac{D_{x_r,t} + \frac{\frac{D_t}{D}+D}{1+D}}{D_{x_r,t} + \left(\frac{\frac{D_t}{D}+D}{1+D}\right)^2},
\end{aligned} \tag{58}$$

which is a function of $l_{x_r,t}$ and D_t exclusively. If $D_t > D$ naturally yields $\beta_t^\star < 1$. Of course, if $D_t = D$ the expression (58) yields $\beta_t^\star = 1$. Finally, Equation (34) further simplifies to:

$$
\begin{aligned}
\widetilde{\delta_t^\star} &= \delta_t^\star \frac{1 - \frac{\alpha_t}{\delta_t^\star} + \left(\frac{\alpha_t}{\delta_t^\star}\right)^2 \frac{1}{1-l_{x_r,t}}}{l_{x_r,t} + \left(\frac{\alpha_t}{\delta_t^\star}\right)^2 \frac{1}{1-l_{x_r,t}}} = \delta_t^\star \frac{1 - \frac{\tilde{\delta}}{\delta_t^\star}(1-l_{x_r,t}) + \left(\frac{\tilde{\delta}}{\delta_t^\star}\right)^2 (1-l_{x_r,t})}{l_{x_r,t} + \left(\frac{\tilde{\delta}}{\delta_t^\star}\right)^2 (1-l_{x_r,t})} \\
&= \delta_t^\star \frac{\frac{1}{1-l_{x_r,t}} - \frac{\tilde{\delta}}{\delta_t^\star} + \left(\frac{\tilde{\delta}}{\delta_t^\star}\right)^2}{D_{x_r,t} + \left(\frac{\tilde{\delta}}{\delta_t^\star}\right)^2} = \delta_t^\star \frac{\frac{1}{1-l_{x_r,t}} - \frac{\frac{D_t}{D}+D}{1+D} + \left(\frac{\frac{D_t}{D}+D}{1+D}\right)^2}{D_{x_r,t} + \left(\frac{\frac{D_t}{D}+D}{1+D}\right)^2} \\
&= \tilde{\delta} \frac{1+D}{\frac{D_t}{D}+D} \frac{\frac{1}{1-l_{x_r,t}} - \frac{\frac{D_t}{D}+D}{1+D} + \left(\frac{\frac{D_t}{D}+D}{1+D}\right)^2}{D_{x_r,t} + \left(\frac{\frac{D_t}{D}+D}{1+D}\right)^2},
\end{aligned} \tag{59}
$$

which, again, is an expression that solely depends on $l_{x_r,t}$ and D_t. If $D_t = D$, Equation (59) simplifies to

$$
\widetilde{\delta_t^\star} = \tilde{\delta} \frac{\frac{1}{1-l_{x_r,t}} - 1 + 1^2}{D_{x_r,t} + 1^2} = \tilde{\delta} \frac{\frac{1}{1-l_{x_r,t}}}{\frac{l_{x_r,t}}{1-l_{x_r,t}} + 1} = \tilde{\delta} \frac{\frac{1}{1-l_{x_r,t}}}{\frac{l_{x_r,t}+1-l_{x_r,t}}{1-l_{x_r,t}}} = \tilde{\delta}.
$$

5 Numerical application

We apply the obtained results for both proposed risk sharing levels: sharing of the demographic risk between the workers and the retirees and, further, sharing of this risk between the different generations of retirees. Therefore, we begin with the calibration of the dependency ratio process on data of the Belgian population.

5.1 Dependency ratio

We consider that the dependence ratio process D follows a Black-Karasinski model (Black and Karasinski, 1991) with a constant reverting value. With this model, the logarithm of the dependence ratio follows a mean-reversion process

$$
\mathrm{d} \ln D_t = \alpha \left(\ln D_\infty - \ln D_t\right) \mathrm{d}t + \sigma \, \mathrm{d}W_t
$$

where α, D_∞ and σ are strictly positive constants and W_t is a Brownian motion. α is the mean reversion rate, D_∞ is the long term mean reverting value of the dependence ratio and σ is the instantaneous volatility. With this model, the dependence ratio follows a log-normal distribution and is strictly positive, as expected for this ratio.

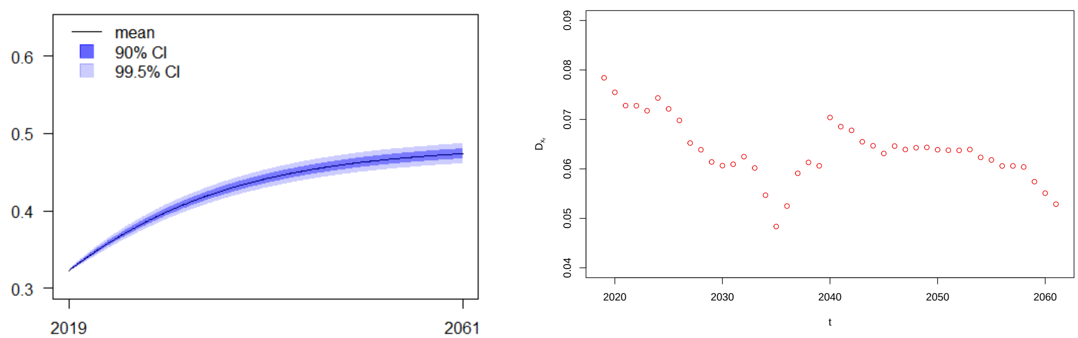
The dependency ratio in the Black-Karasinski model satisfies the following stochastic differential equation

$$
\mathrm{d}D_t = \left(\alpha \ln D_\infty + \frac{\sigma^2}{2} - \alpha \ln D_t\right) D_t \mathrm{d}t + \sigma D_t \mathrm{d}W_t
$$

whose the solution is

$$
D_t = \exp\left(\ln D_0 \, e^{-\alpha t} + \ln D_\infty \left(1 - e^{-\alpha t}\right) + \sigma \int_0^t e^{-\alpha(t-u)} \, \mathrm{d}W_u\right) .
$$

Figure 4: Projection of the dependency ratio under Black-Karasinski and D_{x_r} proportion of new retirees with respect to all retired population



(a) D_t (b) D_{x_r}

This process follows a log-normal distribution

$$D_t \sim \text{LogN}\left(\mu_t, \sigma_t^2\right)$$

with mean and variance given by:

$$\begin{aligned} \text{E}[D_t] &= \exp\left(\mu_t + \frac{\sigma_t^2}{2}\right) \\ \text{Var}[D_t] &= \exp\left(2\,\mu_t + \sigma_t^2\right)\left(e^{\sigma_t^2} - 1\right) \end{aligned}$$

with

$$\begin{aligned} \mu_t &= \ln D_0\, e^{-\alpha t} + \ln D_\infty\left(1 - e^{-\alpha t}\right) \\ \sigma_t^2 &= \frac{\sigma^2}{2\alpha}\left(1 - e^{-2\alpha t}\right) \,. \end{aligned}$$

When $t = \infty$, the drift μ_t and volatility σ_t^2 become:

$$\begin{aligned} \mu_\infty &= \ln D_0\, e^{-\alpha\infty} + \ln D_\infty\left(1 - e^{-\alpha\infty}\right) = \ln D_\infty \\ \sigma_\infty^2 &= \frac{\sigma^2}{2\alpha}\left(1 - e^{-2\alpha\infty}\right) = \frac{\sigma^2}{2\alpha} \,. \end{aligned}$$

And the asymptotic mean and variance are

$$\begin{aligned} \lim_{t\to\infty} \text{E}[D_t] &= \exp\left(\mu_\infty + \frac{\sigma_\infty^2}{2}\right) = \exp\left(\ln D_\infty + \frac{\sigma^2}{4\alpha}\right) \\ \lim_{t\to\infty} \text{Var}[D_t] &= \exp\left(2\,\mu_\infty + \sigma_\infty^2\right)\left(e^{\sigma_\infty^2} - 1\right) \\ &= \exp\left(2\,\ln D_\infty + \frac{\sigma^2}{2\alpha}\right)\left(\exp\left(\frac{\sigma^2}{2\alpha}\right) - 1\right) \,. \end{aligned}$$

Table 2: Target values for S1–S4 when $D_0 = 0.32$, $D_\infty = 0.48$, $\delta_0 = 0.5$ and $\pi_0 = 0.16$.

Scenario	DB	Canonical	Musgrave	DC
ρ	0	$\frac{1}{2}$	$\frac{1}{2}$	1
$\bar{\delta}$	0.50	0.50	0.39	0.50
$\bar{\pi}$	0.16	0.16	0.19	0.16
$\bar{D}$	0.32	0.32	0.48	0.32

We work on data of the annual projection of the Belgian population[9] (from 2019 to 2061) to estimate the projection of the dependence ratio. We suppose a constant retirement age and a complete career (from 20 to 65 years) for each worker. We calibrate our model with the least squares method and we obtain the following parameters for the mean reversion rate $\alpha = 0.059$, the long term mean value $D_\infty = 0.47$ and the instantaneous volatility $\sigma = 0.0046$. The maximum likelihood calibration provides very similar results. Figure 4 presents the projection of the dependency ratio and its 90 % and 99.5% confidence intervals and the proportion of new with respect to all retirees. The *baby-boom* effect is clearly present until 2045.

5.2 1st level: Risk-sharing between workers and retirees

For the risk sharing between the workers and the retirees, we analyse the impact of this dependency ratio process on the contributions of the workers and the benefits of the retirees trough respectively the increase of the contribution rate π and the decrease of the benefit ratio δ. These both variables are determined by the optimisation proposed on Section 4.1.

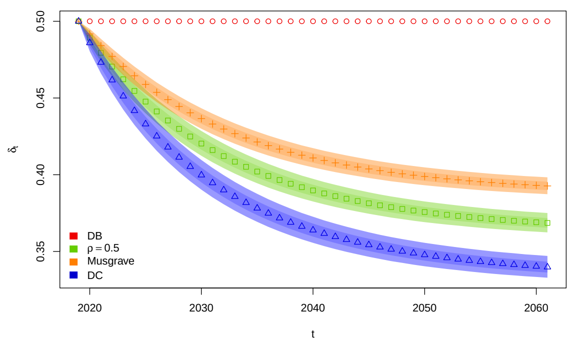
We study scenarios that align with those shown in Table 1 albeit with parameters that fit our Belgian data. Indeed, since $D_{2019} = D_0 = 0.32$ and $\delta_0 = 0.5$ we obtain $\pi_0 = 0.16$ through the PAYG equilibrium Equation (5). We perform our calibration and basic projection until 2061, last available date in our calibration set. We denote in what follows *canonical* the combination of the *canonical* target choice of $\bar{\delta} = \delta_0$ and $\bar{\pi} = \pi_0$ together with the democratic weight parameter $\rho = \frac{1}{2}$. The DB, DC and *canonical* scenarios all share the same targets and only differ in the weight parameter ρ. The Musgrave scenario, on the other hand, is selected using the same rationale as S4 in Example 1 but with our calibrated data. Indeed, $\bar{\delta} = 0.39$ and $\bar{\pi} = 0.19$ that result from π_∞ and δ_∞ from Equations (14) and (13) when replacing $D_t = D_\infty = D_{2061} = 0.48$ under the Musgrave rule for a choice of $M = 0.4773$. This choice yields initial equivalence between the different plans, ensuring comparability.

Figure 5 shows the mean and confidence intervals of the optimal benefit ratio $\delta_t^\star$ (22) (Subfigure 5(a)) and contribution rate $\pi_t^\star$ (23) (Subfigure 5(b)) for scenarios in Table 2. In contrast to the results shown in Example 1, we made parameter choices to ensure starting at the same benefit and contribution level in 2019. The results obtained, while variable, exhibit tight confidence intervals, as expected. With the dependency ratio's low volatility and anticipation of a smooth transition to an aged society, little deviation from the predetermined path is expected.

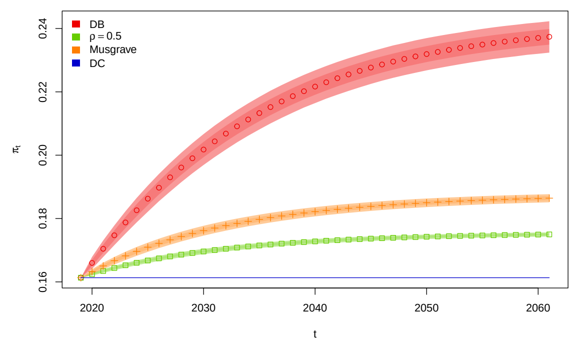
The benefit ratio shows comparable variability, whereas a higher risk can be detected in the contribution rate. The variability of the *canonical* or Musgrave rule is limited, whereas the contribution rate of the DB case has wider bounds with up to $\pm 0.5\%$ absolute deviation

[9] Belgian Federal Planning Bureau.

Figure 5: Mean and confidence intervals of the optimal benefit ratio $\delta_t^\star$ (22) and contribution rate $\pi_t^\star$ (23) for scenarios in Table 2.



(a) $\delta_t^\star$



(b) $\pi_t^\star$

Notes: Confidence intervals depicted are (0.5%,99.5%) [light] and (1%,99%) [dark].

from the mean. Clearly, for the benefit ratio of the DB, it becomes deterministic, as well as the contribution rate of the DC.

Let us initially focus on the DB, *canonical*, and DC scenarios. Despite targeting the same long-term $\bar{\delta}$, the choice of ρ of 0, $\frac{1}{2}$, and 1 respectively yield substantially different benefit levels in both the short and long term. Since $D_t > D_0 = 0.32$, we observe that $\delta_t^\star < \bar{\delta}$ per the relationship shown in Equation (29). The level of $\bar{\delta} = 0.5$ is only attained in the pure DB scenario, as expected. In the DB case, we obtain a fixed level of $\delta_t^\star$ for all t at the expense of a contribution level that increases from $\pi_{2019} = 0.16$ to $\pi_{2060} = 0.24$ at the end of our projection horizon. On the other hand, in the DC scheme, we fix the contribution rate at the initial level and reduce the aggregate benefit rate from 0.50 to approximately 0.35. Our *canonical* choice yields an intermediate solution with a higher long-term δ_t than in a pure DC scenario, with a contribution rate increase from 16% to 17.5% over a 40-year horizon. We believe such an increase would be bearable even in high-tax countries.

Now, let us shift our focus to the Musgrave rule. It has been calibrated to have the same initial conditions, but given its structure, it necessarily has a different long-term target $\bar{\pi}$ and $\bar{\delta}$. We target a long-term contribution rate of $\bar{\pi} = 0.19$ and a benefit ratio $\bar{\delta} = 0.39$, and these values are exactly attained at the end of our simulation since $\bar{D} = D_\infty$. Again, per the dichotomy shown in Equation (29), $\delta_t^\star > \bar{\delta}$ for all t since $\bar{D} > D_t$ for all t. Note that, despite the choice of targets, the benefit ratio (contribution rate) will decrease (increase) in all scenarios. The level of decrease will primarily depend on the long-term financing capacity through $\bar{\delta}$. If we allow the system to increase the contribution rate to 19%, as in the Musgrave case, then, by PAYG equilibrium (6), a benefit ratio of 39% will be attained. On the other hand, if we want to either keep finances or benefits fixed, aggregate benefit levels will have to decrease (DC) or contributions will have to increase (DB) substantially.

We would like to briefly compare the effect of the targets for a given weight parameter ρ. The *canonical* and Musgrave scenarios both have the demographic weight $\rho = \frac{1}{2}$ but different targets. This is natural as the demographic weight simply indicates that equal weight is given between the variation allowed with respect to the targets. If the targets differ, the trajectories towards that target will differ as well. However, it is clear that these two scenarios provide similar outcomes, both in benefit and contribution rate evolution.

Finally, we would like to highlight that δ_t always decreases as our population ages. This means that, on a relative level, lower pensions are paid out compared to current salaries. However, this does not imply, as shown in Subsection 5.3, that the nominal level of pensions will decrease over time.

5.3 2nd level: Risk sharing between the different generations of retirees

In the second level, we are interested in sharing the effect of a decreasing aggregate benefit ratio δ_t among new and old retirees through $\tilde{\delta}_t$ and $\tilde{\beta}_t$, respectively. We assume that our targets correspond to $\overline{\overline{\delta}}_t = \delta_t$ and $\overline{\beta}_t = 1$ as indicated in Equation (43). According to Remark 5, our optimal replacement rate when $\eta = 0$ corresponds to $\overline{\overline{\delta}}_t$, while $\eta = 1$ yields perfect wage indexation with $\tilde{\beta}_t = 1$. It is noteworthy that a general family of risk-sharing mechanisms, depending on η, arises after obtaining the first-level optima for each of our four scenarios. Furthermore, we assume that the proportion of newly retired individuals to the total retiree population corresponds to the projected population distribution given by the Belgian Federal Planning Bureau.

Figure 6 shows the mean and confidence intervals of the optimal replacement rate $\tilde{\delta}_t^\star$ (45) and optimal yearly sustainability factor $\beta_t^\star$ (44) for $\eta = 0,\ 0.5,\ 0.95, 1$ for scenarios in Table 2. A quick look at the Figure shows that the variability of our optima is greater than in the 1st level, especially regarding the yearly sustainability factor for $\eta < 1$. Obviously, for $\eta = 1$, we obtain total wage indexation and hence no uncertainty. Similarly, the DB scenario yields a constant benefit ratio and full indexation. As seen in Figure 5, this is only possible since the contribution rate increases substantially over the studied horizon.

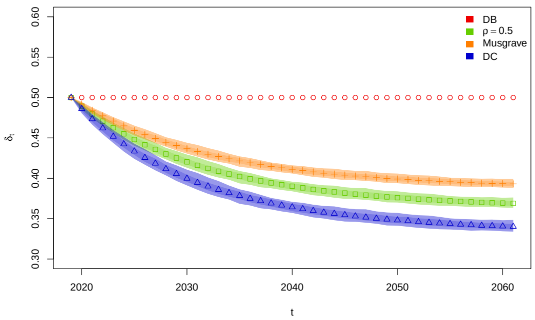
Now, let us analyze the global trends of $\tilde{\beta}_t$ for $\eta < 1$. We observe that the sustainability factor decreases significantly when D_t deviates for the first time from the steady-state past. As illustrated in Example 3, a small change in the first-level δ_t results in a substantial change in the sustainability factor. Moreover, our population distribution anticipates a substantial influx of retirees until 2040, corresponding to the *baby-boom* generation (Figure 4(b)). The combination of these factors leads to a considerable initial decrease in the sustainability ratio, aiming to maintain a reasonable level of benefits for all retirees.

However, as the first level yields increasingly smaller δ_t as a consequence of aging and pensions globally start to decrease relative to wages, higher indexation becomes affordable. By the end of our projection horizon, the sustainability factors in the *canonical*, Musgrave, and DC converge towards 1, the same level as the DB scheme, irrespective of the long-term targets $\bar{\beta}_t$ and $\bar{\pi}_t$. Furthermore, similar to Baurin and Hindriks (2023), we observe that a small decrease in indexation is sufficient to maintain the sought equilibrium.

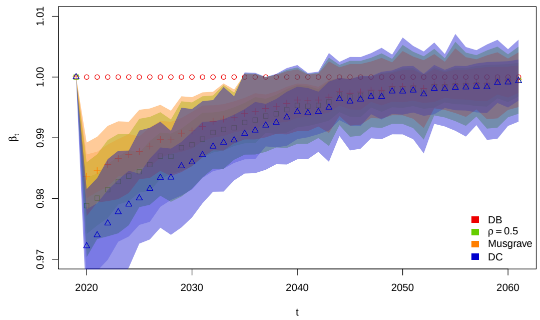
Now, let us shift our focus to η values closer to or equal to 1. We observe that for this parameter, the benefit ratio, which follows a steady decreasing trend for other η values, starts to exhibit erratic behavior. As we approach $\eta = 1$, full indexation of older retirees' pensions becomes a priority. Consequently, the replacement rate is determined based on the remaining financing capacity after providing full indexation to existing retirees. For instance, when $\eta = 0.95$, there is a one-time decrease from 50% to 37% between 2019 and 2020. Due to a substantial and steady influx of new retirees, the pension system undergoes further corrections, resulting in replacement rates dropping below 30%. Despite these significant adjustments, the system can still afford high indexation, albeit on much lower pensions for those who have recently retired.

After reaching a record low in 2025, the replacement rate starts to increase, reaching values

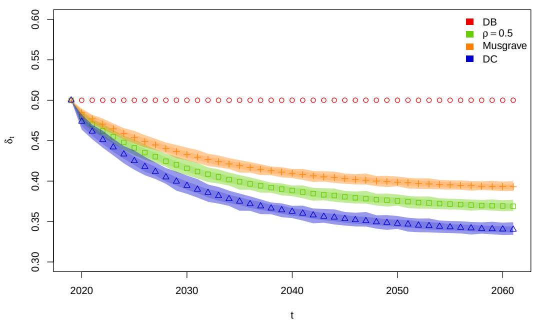
Figure 6: Mean and confidence intervals of the optimal replacement rate $\tilde{\delta}_t^\star$ (45) and optimal yearly sustainability factor $\beta_t^\star$ (44) for $\eta = 0,\ 0.5,\ 0.95, 1$ for scenarios in Table 2.



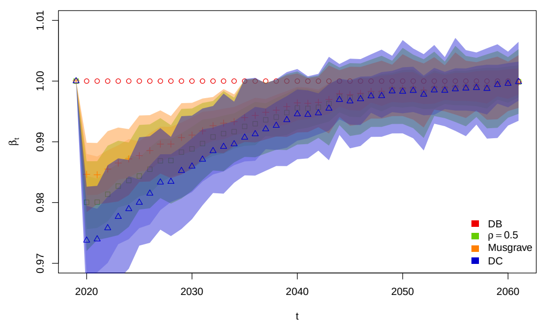
(a) $\tilde{\delta}_t^\star$, $\eta = 0$



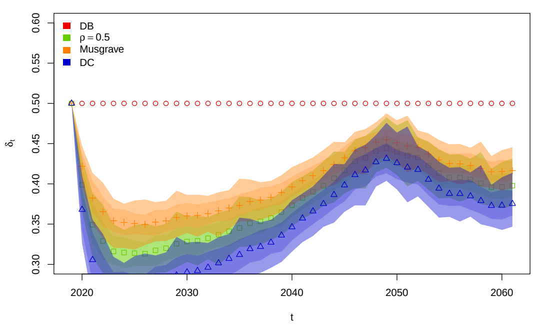
(b) $\beta_t^\star$, $\eta = 0$



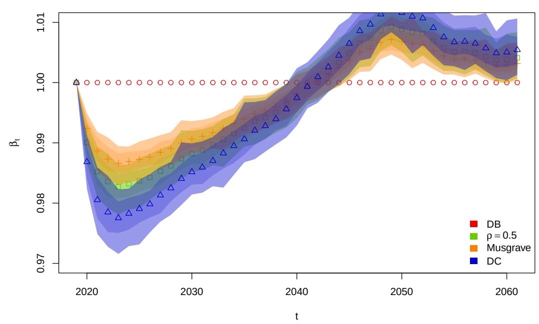
(c) $\tilde{\delta}_t^\star$, $\eta = 0.5$



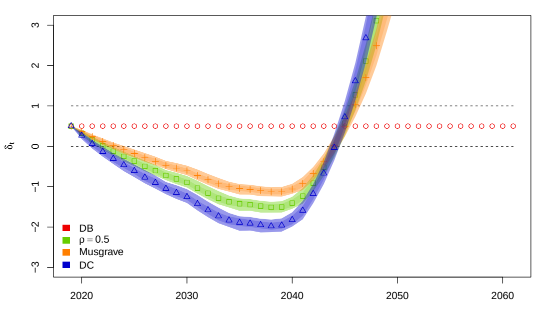
(d) $\beta_t^\star$, $\eta = 0.5$



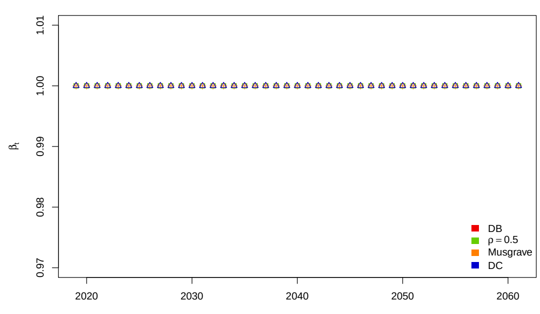
(e) $\tilde{\delta}_t^\star$, $\eta = 0.95$



(f) $\beta_t^\star$, $\eta = 0.95$



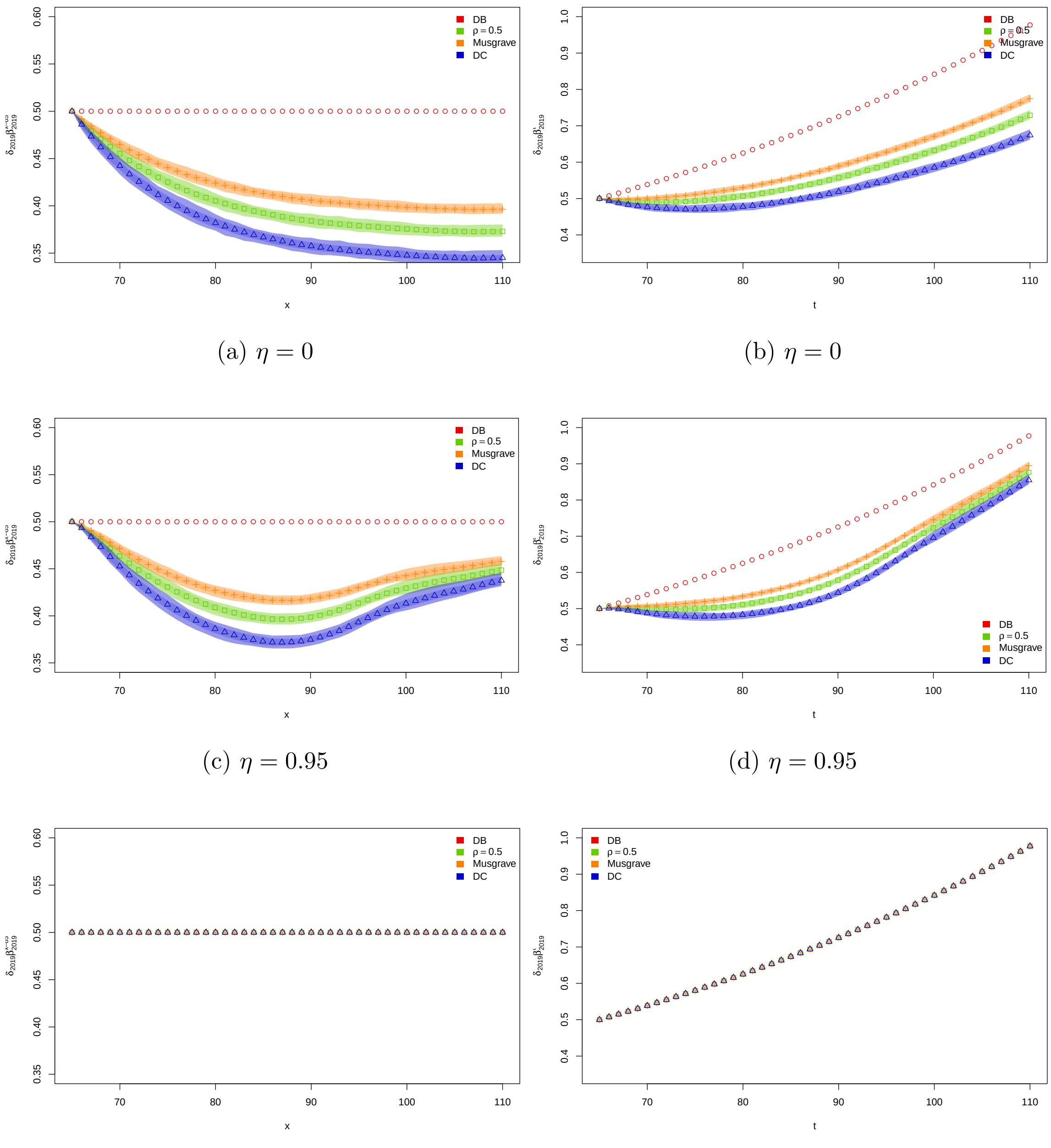
(g) $\tilde{\delta}_t^\star$, $\eta = 1$



(h) $\beta_t^\star$, $\eta = 1$

Notes: Confidence intervals are (0.5%,99.5%) and (1%,99%).

Figure 7: Indexed pension with adjustments $\tilde{\delta}_{2019}\beta_{2019}^{x-65}(1+g)^{x-65}$ (mean and CI) for the cohort retiring in 2019 for $\eta = 0, 0.95, 1$ for scenarios in Table 2. Scenarios with no wage growth ($g = 0\%$) and wage growth ($g = 1.5\%$).



(a) $\eta = 0$ (b) $\eta = 0$

(c) $\eta = 0.95$ (d) $\eta = 0.95$

(e) $\eta = 1$ (f) $\eta = 1$

Notes: Confidence intervals are (0.5%,99.5%) and (1%,99%).

close to 45% in 2050 in the *canonical*, Musgrave, and DC scenarios before decreasing again. This increase, despite the aging environment, is driven by the sustainability factor exceeding 1, made possible by the very low new pensions paid during the *baby-boom* shock. Once the sustainability factor surpasses 1, the pension volume starts to exceed what is affordable, and new replacement rates need to decrease again. This trend becomes more evident when $\eta = 1$, and full indexation is promised. In this case, we even observe negative replacement rates after 3 years. Guaranteeing full wage indexation to all existing retirees, regardless of the new retirees' benefits, comes at a high cost. In this scenario, the replacement rate remains negative until 2043 when it turns positive again. These aberrant low and even negative new pensions lead, at a certain moment, to aberrant high replacement rates as the funding capacity of the state is substantial. It has been paying negative pensions for over 20 years, and when the inflow of the *baby-boom* generation stops, there is suddenly such a surplus that it can even afford over a 300% replacement rate.

In summary, guaranteeing full indexation is simply too expensive and unreasonable in an aging society experiencing a demographic shock like the *baby-boom*. On the other hand, we observe that a small decrease in indexation ($\tilde{\beta}_t < 1$) guarantees global replacement rate to new retirees and and aggregate benefit ratio stability. The only downside is that, despite the low volatility observed in the first level, the values on a year-to-year basis are less predictable and have wider bounds. Nevertheless, the maximum variability amounts to $\pm 2\%$.

Finally, we highlight that despite sustainability factors become lower than 1, pensions are very likely to increase on a nominal basis for a representative retiree. Figure 7 shows the mean and confidence interval of the indexed pension $\tilde{\delta}_{2019}\beta_{2019}^{x-65}(1+g)^{x-65}$ with adjustments for the cohort retiring in 2019 for $\eta = 0, 0.95, 1$ for scenarios in Table 2. The scenario $\eta = 0.5$ is omitted as it yields similar values to $\eta = 0$, as given in Figure 6. Two hypotheses for the wage growth are shown: $g = 0\%$, indicating no wage growth in the general economy, and $g = 1.5\%$. We choose the cohort retiring in 2019 as it is the one that will bear the highest effect of the *baby-boom* generation depicted in Figure 4(b) and is the first one to move out of the relative steady-state.

Subfigures 7(a), 7(c), and 7(e) show the effect when no wage growth is present. For $\eta = 0$, the replacement rate is equal to the aggregate benefit ratio, which at this stage is still 50%. Subsequent $\tilde{\beta}_t < 1$ yields, unfortunately, a decreasing purchasing power of pensions over time, despite the initially high replacement rate. When $\eta = 0.95$, the initially lower and then higher indexation provided over time (Subfigure 6(f)) allows for an increasing nominal trend despite no wage growth. However, this growth is attained after reaching the age of 90, so few retirees from the 2019 cohort would be able, under this scenario, to benefit from the increasing nominal pensions. When $\eta = 1$ and full indexation is guaranteed, we observe that the cohort would receive a fixed nominal pension level throughout their lifetime under all scenarios from Table 2. Of course, this results in very high deficits and negative replacement rates for new subsequent retirees, as depicted in Figure 6(g).

No wage growth, although realistic for short periods of time, is neither a desirable nor a realistic assumption. If wage growth is supposed to be 1.5%, we observe that pensions for all η values grow over time, albeit at a lower pace than in a classical (but expensive for the workers) DB scenario. Nominal pensions would increase from 0.5 to 0.8 over time under the Musgrave rule and to 0.7 for the DC scheme. The *canonical* would yield around 0.75 for the considered cohort.

6 Conclusion

Our study focuses on two levels of risk-sharing in the context of a social security pay-as-you-go pension system. In the first level, we analyze the impact of demographic risk, emphasizing the dependency ratio, measuring the proportion of retirees to the working-age population.

We develop an optimization scheme based on Cairns (2000), penalizing deviations from a pre-specified target with a weight ρ characterizing intermediate systems between DB and DC. We demonstrate mathematically that the long-term evolution of the contribution rate and benefit ratio is primarily influenced by the mismatch between the observed dependency ratio D_t and the ratio implied by our chosen long-term targets $\bar{D}$.

To assess uncertainty, we model the dependency ratio with a Black-Karasinski model incorporating mean reversion observed in empirical Belgian data. We find tight confidence intervals for contribution and benefit ratios. Considering four risk-sharing rules (DB, DC, canonical, Musgrave), we observe a consistent decrease in the benefit ratio as our population ages, implying lower pensions relative to current salaries. However, the actual amount paid depends on the chosen risk-sharing mechanism; for example, higher pensions under a DB scenario would require a 50% increase in contribution rates.

In the second level, the focus shifts to risk-sharing between different generations of retirees, studying the replacement rate for new retirees and the sustainability factor that multiplies salary-linked pension indexation affecting all existing retirees. Similar to the first level, we develop a general family of risk-sharing mechanisms identified by the weight parameter η. Our analysis reveals the offering full salary-linked indexation to retirees and affecting only the replacement rate of new retirees is often unrealistic, especially during demographic shocks like the *baby-boom*. Compared to the first level, the confidence interval of sustainability factors is greater, indicating a policy mechanism to share remaining risk after fixing benefit levels. A slight decrease in indexation provides stability in replacement rates and benefit ratios, offering a more feasible approach to demographic challenges. Despite sustainability factors falling below 1, we show that nominal pensions are likely to increase for retirees, ensuring financial support in an aging society.

With this in mind, we identify a few avenues for future research. Our paper focuses on a pure pay-as-you-go system that is self-financing, with current contributions expected to cover current pension expenditures. A natural extension involves exploring the role of funding elements invested in financial markets to alleviate the impact of the increasing dependency ratio. Secondly, our optimization scheme relies on long-term targets and a specific time horizon, which is justified given the primary aim of mitigating the effect of a known, relatively stable demographic trend associated with overall aging and the *baby-boom* shock. A potential area for future research could involve exploring time-consistent strategies that incorporate the complexity of considering the welfare of multiple (current and future, yet-to-be-born) generations, independent of a pre-specified horizon.

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