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Abstract

Systematic improvements in mortality results in dependence in the survival distributions of insured lives. This is not allowed for in standard life tables and actuarial models used for annuity pricing and reserving. Systematic longevity risk also undermines the law of large numbers; a law that is relied on in the risk management of life insurance and annuity portfolios. This paper applies a multivariate gamma distribution to incorporate dependence. Lifetimes are modelled using a truncated multivariate gamma distribution that induces dependence through a shared gamma distributed component. Model parameter estimation is developed based on the method of moments and generalized to allow for truncated observations. The impact of dependence on the valuation of a portfolio, or cohort, of annuitants with similar risk characteristics is demonstrated by applying the model to annuity valuation. The dependence is shown to have a significant impact on the risk of the annuity portfolio as compared with traditional actuarial methods that implicitly assume independent lifetimes.

Keywords: systematic longevity risk, dependence, multivariate gamma, lifetime distribution, annuity valuation

JEL Classifications: G22, G32, C13, C02

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1 Introduction

Systematic improvements in mortality results in dependence in the survival distributions of insured lives. This is not allowed for in standard life tables and actuarial models used for annuity pricing and reserving. Systematic longevity risk also undermines the law of large numbers; a law that is relied on in the risk management of life insurance and annuity portfolios. Given recent world-wide trends by employers towards the elimination of pension scheme liabilities, understanding systematic longevity risk is especially relevant for bulk annuity providers; see e.g. Hull (2009).

This paper applies a multivariate gamma distribution to model dependent lifetimes within a pool of individuals. Lifetimes are often modelled with parametric distributions such as the gamma distribution, which has similar properties to the Weibull distribution; see e.g. Klein and Moeschberger (1997). Dependence between the lifetimes is captured with a common stochastic component. The multivariate dependence structure is developed from the trivariate reduction method used to generate two dependent random variables from three independent random variables. This trivariate method was used to generate the bivariate version of the multivariate gamma distribution in Chereiyan (1941). The method uses the fact that the sum of gamma random variables with the same rate parameter also follows a gamma distribution with that same rate parameter. The trivariate method was generalized to multivariate reduction and the bivariate gamma distribution model extended to the multivariate setting by Ramabhadran (1951) and applied by Mathai and Moschopoulus (1991), and Chatelain *et al.* (2006), amongst others. The multivariate gamma distribution has found many applications in actuarial science including Furman and Landsman (2005).

The paper develops estimation theory for the multivariate gamma distribution in the presence of truncation, which we highlight as the main theoretical contribution to the literature. To quantify the effect of dependence, life annuities are valued with the model and compared to standard actuarial models for annuity pricing. Although the expected present value of the annuity payment streams do not vary much as the level of dependence increases, the variance increases significantly more than the square root of the size of the portfolio for the independence case. Risk based capital reflects the variance of the payment stream and the cost of this capital is reflected in the market pricing of annuities. Hence, we provide evidence that dependence is a significant factor with important implications for annuity pricing and risk based capital, which we highlight as the main practical contribution. The model presented here provides a tractable method for estimating the dependence and computing the distribution of life annuity values. Finally, an assessment of the model fit to data is provided based on Norwegian population mortality. Some insight is provided on ways in which the fit can be improved, the implementation of which is anticipated in future research.

Organization of the paper: Section 2 defines the multivariate gamma dependence structure for survival models for a pool of lives. Section 3 provides the estimation of the parameters of the model by method of moments. We consider the case when samples are given both with and without truncation. The former is essentially more complicated, but required in practice. The performance of the estimation methods are assessed with simulation. Section 4 outlines the application to survival theory including implications for annuity values and portfolio risk based on standard deviation of values. Section 5 reports the fitting of the model to Norwegian population data. Section 6 concludes the paper.

2 Multivariate Gamma Survival Model

The model is applied to individuals within a pool of lives. We assume M pools, each constituted of N lives. The pools can, in general, be of individuals with the same age or other characteristics that share a common risk factor. Let $T_{i,j}$ be the survival time of individual $i \in \{1, \dots, N\}$ in pool $j \in \{1, \dots, M\}$. We assume the following model for the individual lifetimes:

$$T_{i,j} = \frac{\alpha_0}{\alpha_j} Y_{0,j} + Y_{i,j},$$

where

- $Y_{0,j}$ follows a gamma distribution with shape parameter γ_0 and rate parameter α_0, $G(\gamma_0, \alpha_0)$, $j \in \{1, \dots, M\}$,
- $Y_{i,j}$ follows a gamma distribution with shape parameter γ_j and rate parameter α_j, $G(\gamma_j, \alpha_j)$, $i \in \{1, \dots, N\}$ and $j \in \{1, \dots, M\}$,
- The $Y_{i,j}$ are independent, $i \in \{0, \dots, N\}$ and $j \in \{1, \dots, M\}$.

Hence, there is a common component $Y_{0,j}$ within each pool j that impacts the survival of the individuals of that pool (i.e. $Y_{0,j}$ captures the impact of systematic mortality dependence between the lives in pool j). The parameters γ_j and α_j can jointly be interpreted as the risk profile of pool j.

From the properties of the gamma distribution it immediately follows that the survival times $T_{i,j}$ are also gamma distributed with shape parameter $\tilde{\gamma}_j = \gamma_0 + \gamma_j$ and rate parameter α_j. One can see that within each pool, individual lifetimes are dependent and all follow the same gamma distribution, $G(\tilde{\gamma}_j, \alpha_j)$.

3 Parameter Estimation

In this section we consider parameter estimation using the method of moments. For an excellent reference we can suggest, for example Lindgren (1993) (Ch. 8, Theorem 6).

Notation

Before we undertake parameter estimation, we provide some necessary notation concerning raw and central, theoretical and sample, moments. Consider arbitrary random variable X. We denote with $\alpha_k(X)$ and $\mu_k(X)$ the k^{th}, $k \in \mathbb{Z}^+$, raw and central (theoretical) moments of X, respectively. That is,

$$\begin{aligned} \alpha_k(X) &= E[X^k], \\ \mu_k(X) &= E[(X - \alpha_1(X))^k]. \end{aligned}$$

Next, consider random sample $\mathbf{X} = (X_1, \ldots, X_n)'$. The raw sample moments are given by

$$a_k(\mathbf{X}) = \frac{1}{n}\sum_{i=1}^{n} X_i^k, \quad k \in \mathbb{Z}^+.$$

For $X_1, \ldots, X_n$ identically distributed, the raw sample moments are unbiased estimators of the corresponding raw moments of X_1:

$$E[a_k(\mathbf{X})] = \alpha_k(X_1).$$

Finally, we define the *adjusted* second and third central sample moments as

$$\begin{aligned} \widetilde{m}_2(\mathbf{X}) &= \frac{1}{n-1}\sum_{i=1}^{n}(X_i - a_1(\mathbf{X}))^2, \\ \widetilde{m}_3(\mathbf{X}) &= \frac{n}{(n-1)(n-2)}\sum_{i=1}^{n}(X_i - a_1(\mathbf{X}))^3. \end{aligned}$$

For $X_1, \ldots, X_n$ independent and identically distributed, these (adjusted) central sample moments are unbiased and consistent estimators of the corresponding central moments of X_1:

$$E[\widetilde{m}_2(\mathbf{X})] = \mu_2(X_1) \qquad \text{and} \qquad E[\widetilde{m}_3(\mathbf{X})] = \mu_3(X_1).$$

3.1 Parameter Estimation for Lifetime Observations

We assume we are given samples, $\mathbf{T}_1, \ldots, \mathbf{T}_M$, from the pools, where $\mathbf{T}_j = (T_{1,j}, \ldots, T_{N,j})'$. In order to facilitate estimation, we presently make the assumption that $\alpha_0 = 1$. This assumption is equivalent to setting the rate parameter of the systematic component, $(\frac{\alpha_0}{\alpha_j} Y_{0,j})$, equal to that of the idiosyncratic component, $(Y_{i,j})$, within each pool j.

We begin by considering the $\mathbf{T}_j$ separately in order to estimate corresponding parameters γ_j and α_j, as well as predict the value of $Y_{0,j}$. Subsequently, we combine the obtained predictions of $Y_{0,1}, \ldots, Y_{0,M}$ in order to estimate γ_0.

In our estimation procedure, we utilize the first raw sample moment and the second and third central sample moments. Define $\mathbf{Y}_j = (Y_{1,j}, \ldots, Y_{N,j})'$. For the first raw sample moment, we obtain

$$a_1(\mathbf{T}_j) = \frac{1}{N} \sum_{i=1}^{N} T_{i,j} = \frac{1}{N} \sum_{i=1}^{N} \frac{1}{\alpha_j} Y_{0,j} + \frac{1}{N} \sum_{i=1}^{N} Y_{i,j} = \frac{1}{\alpha_j} Y_{0,j} + a_1(\mathbf{Y}_j). \tag{1}$$

For the second central sample moment, we obtain

$$\begin{aligned}
\widetilde{m}_2(\mathbf{T}_j) &= \frac{1}{N-1} \sum_{i=1}^{N} (T_{i,j} - a_1(\mathbf{T}_j))^2 \\
&= \frac{1}{N-1} \sum_{i=1}^{N} \Big(\frac{\alpha_0}{\alpha_j} Y_{0,j} + Y_{i,j} - \frac{\alpha_0}{\alpha_j} Y_{0,j} - a_1(\mathbf{Y}_j) \Big)^2 \\
&= \frac{1}{N-1} \sum_{i=1}^{N} (Y_{i,j} - a_1(\mathbf{Y}_j))^2 = \widetilde{m}_2(\mathbf{Y}_j).
\end{aligned}$$

Similarly, for the third central sample moment, we obtain

$$\widetilde{m}_3(\mathbf{T}_j) = \widetilde{m}_3(\mathbf{Y}_j).$$

We take expectations of our sample moments in order to formulate a system of equations. Since each pool contains only one realization from the $G(\gamma_0, \alpha_0 = 1)$ distribution, namely, $Y_{0,j}$, it is not prudent to take its expected value. Therefore, we condition on $Y_{0,j}$. Since $Y_{1,j}, \ldots, Y_{N,j}$ are identically distributed, the first raw sample moment is an unbiased estimator of the first raw moment of $Y_{1,j}$. Consequently, we have

$$E[a_1(\mathbf{T}_j)|Y_{0,j}] = \frac{1}{\alpha_j} Y_{0,j} + E[a_1(\mathbf{Y}_j)] = \frac{1}{\alpha_j} Y_{0,j} + \alpha_1(Y_{1,j}) = \frac{1}{\alpha_j} Y_{0,j} + \frac{\gamma_j}{\alpha_j}.$$

Furthemore, since $Y_{1,j}, \ldots, Y_{N,j}$ are also independent, the (adjusted) second and third central sample moments are unbiased estimators of the second and third central moments of $Y_{1,j}$, respectively. As a result, we obtain

$$
\begin{aligned}
E[\widetilde{m}_2(\mathbf{T}_j)|Y_{0,j}] &= E[\widetilde{m}_2(\mathbf{Y}_j)] = \mu_2(Y_{1,j}) = \frac{\gamma_j}{\alpha_j^2}, &\quad (2)\\
E[\widetilde{m}_3(\mathbf{T}_j)|Y_{0,j}] &= E[\widetilde{m}_3(\mathbf{Y}_j)] = \mu_3(Y_{1,j}) = \frac{2\gamma_j}{\alpha_j^3}. &\quad (3)
\end{aligned}
$$

Note that the above central sample moments do not depend on $Y_{0,j}$. As a result, equations (2) and (3) can be used to estimate γ_j and α_j. Let us notice that from (1), it follows that for $N \to \infty$,

$$
a_1(\mathbf{T}_j) \xrightarrow{P} \frac{1}{\alpha_j} Y_{0,j} + \frac{\gamma_j}{\alpha_j},
$$

and we cannot estimate parameters of $Y_{0,j}$ from one pool (j-pool). However, the estimators of γ_j and α_j can be substituted into equation (1) to yield a prediction of $Y_{0,j}$. Hence, by replacing the expected sample moments with observed sample moments, we obtain the following:

$$
\begin{aligned}
\widehat{\gamma}_j &= 4\frac{\widetilde{m}_2^3(\mathbf{T}_j)}{\widetilde{m}_3^2(\mathbf{T}_j)},\\
\widehat{\alpha}_j &= 2\frac{\widetilde{m}_2(\mathbf{T}_j)}{\widetilde{m}_3(\mathbf{T}_j)},\\
\widehat{Y}_{0,j} &= a_1(\mathbf{T}_j)\widehat{\alpha}_j - \widehat{\gamma}_j.
\end{aligned}
$$

Finally, we estimate γ_0 using the predicted values of $Y_{0,j}$ and the fact that $E[Y_{0,j}] = \gamma_0$ (since $\alpha_0 = 1$). We obtain

$$
\widehat{\gamma}_0 = \frac{1}{M} \sum_{j=1}^{M} \widehat{Y}_{0,j}.
$$

Summarizing, when considering only one pool j, the parameters γ_j and α_j can be estimated and the random variable $Y_{0,j}$ predicted. In order to estimate γ_0, multiple pools are required.

3.2 Parameter Estimation for Truncated Observations

The results of the previous section cannot be directly used for calibration of parameters of the proposed model, because, in fact, we deal with truncated lifetime data. In this section we consider truncated observations ${}_{\tau_j}T_{i,j} =$

$T_{i,j}|T_{i,j} > \tau_j$ with known truncation point τ_j. As before, we assume $\alpha_0 = 1$. Furthermore, we assume all pools are subject to the same truncation point, that is $\tau_j = \tau$ for all j.

We begin by constructing a useful lemma regarding the raw moments of truncated gamma random variables.

Lemma 1 *Consider $Y \sim G(\gamma, \alpha)$ with probability density and survival function denoted $g(y, \gamma, \alpha)$ and $\overline{Ga}(y, \gamma, \alpha)$, respectively. Define associated truncated random variable ${}_\tau Y = Y|Y > \tau$, where $\tau \geq 0$. The k^{th} raw moment, $k \in \mathbb{Z}^+$, of ${}_\tau Y$ is given by*

$$\alpha_k({}_\tau Y) = \alpha_k(Y) K_k(\tau, \gamma, \alpha),$$

where

$$K_k(\tau, \gamma, \alpha) = \frac{\overline{Ga}(\tau, \gamma + k, \alpha)}{\overline{Ga}(\tau, \gamma, \alpha)}.$$

Proof. The probability density function of ${}_\tau Y$ is given by

$$f_{{}_\tau Y}(y) = \frac{g(y, \gamma, \alpha)}{\overline{Ga}(\tau, \gamma, \alpha)}, \qquad y > \tau.$$

$$\begin{aligned}
\alpha_k({}_\tau Y) &= \frac{\int_\tau^\infty y^k g(y, \gamma, \alpha) dy}{\overline{Ga}(\tau, \gamma, \alpha)} = \frac{\alpha^\gamma \int_\tau^\infty y^k \cdot y^{\gamma-1} e^{-\alpha \cdot y} dy}{\Gamma(\gamma)\overline{Ga}(\tau, \gamma, \alpha)} \\
&= \frac{\alpha^\gamma \Gamma(\gamma + k)\overline{Ga}(\tau, \gamma + k, \alpha)}{\alpha^{\gamma+k}\Gamma(\gamma)\overline{Ga}(\tau, \gamma, \alpha)} = \frac{\Gamma(\gamma + k)}{\Gamma(\gamma)} \frac{1}{\alpha^k} \frac{\overline{Ga}(\tau, \gamma + k, \alpha)}{\overline{Ga}(\tau, \gamma, \alpha)} \\
&= E[Y^k] \frac{\overline{Ga}(\tau, \gamma + k, \alpha_j)}{\overline{Ga}(\tau, \gamma, \alpha)} = E[Y^k] K_k(\tau, \gamma, \alpha) \\
&= \alpha_k(Y) K_k(\tau, \gamma, \alpha).
\end{aligned}$$

■

In the above lemma, K_k can be interpreted as a *truncation adjustment coefficient*, which is required for transforming un-truncated raw moments into truncated raw moments.

We explore the truncated lifetime ${}_\tau T_{i,j}$ by separating it into its component parts: the systematic, un-truncated $\frac{1}{\alpha_j} Y_{0,j}$ and the idiosyncratic, truncated $Y_{i,j}$. We obtain

$${}_\tau T_{i,j} = \frac{1}{\alpha_j} Y_{0,j} + {}_{\tau'} Y_{i,j},$$

where $\tau' = \tau - \frac{1}{\alpha_j}Y_{0,j}$. The truncation on $Y_{i,j}$ must account for the value of the systematic component and hence differs from the relatively simple truncation imposed on $T_{i,j}$.

We first consider the general case and obtain a system of equations that is difficult to solve because of numerical instability. We then consider a simplified structure for which we obtain parameter estimates.

3.2.1 The General Case

In this section, we follow the same method utilized in parameter estimation for un-truncated observations. That is, we aim to use the first raw sample moment, and the second and third central sample moments. Consider given truncated samples ${}_\tau\mathbf{T}_1, \ldots, {}_\tau\mathbf{T}_M$, where ${}_\tau\mathbf{T}_j = ({}_\tau T_{1,j}, \ldots, {}_\tau T_{N,j})'$. From each pool j, we aim to estimate γ_j and α_j, and predict the value of $Y_{0,j}$. Define ${}_{\tau'}\mathbf{Y}_j = ({}_{\tau'}Y_{1,j}, \ldots, {}_{\tau'}Y_{N,j})'$. For the first raw sample moment, we obtain

$$a_1({}_\tau\mathbf{T}_j) = \frac{1}{\alpha_j}Y_{0,j} + \frac{1}{N}\sum_{i=1}^{N} {}_{\tau'}Y_{i,j} = \frac{1}{\alpha_j}Y_{0,j} + a_1({}_{\tau'}\mathbf{Y}_j).$$

For the second and third central sample moment, we obtain

$$\widetilde{m}_2({}_\tau\mathbf{T}_j) = \widetilde{m}_2({}_{\tau'}\mathbf{Y}_j) \quad \text{and} \quad \widetilde{m}_3({}_\tau\mathbf{T}_j) = \widetilde{m}_3({}_{\tau'}\mathbf{Y}_j).$$

Note that $Y_{0,j}$ is present in the truncation point τ'. Hence, unlike in the un-truncated case, we cannot solely use the second and third central moments to estimate γ_j and α_j.

Now suppose that $Y_{0,j}$ is given. Then ${}_{\tau'}Y_{1,j}, \ldots, {}_{\tau'}Y_{N,j}$ are independent and identically distributed. Consequently, the first raw sample moment is an unbiased estimator of $\alpha_1({}_{\tau'}Y_{1,j}|Y_{0,j})$. Moreover,

$$a_1({}_\tau\mathbf{T}_j)|Y_{0,j} \xrightarrow{P} \frac{1}{\alpha_j}Y_{0,j} + \alpha_1({}_{\tau'}Y_{1,j}|Y_{0,j})$$

and the (adjusted) second and third central sample moments are unbiased and consistent estimators of $\mu_2({}_{\tau'}Y_{1,j}|Y_{0,j})$ and $\mu_3({}_{\tau'}Y_{1,j}|Y_{0,j})$, respectively. We take conditional expectations of the sample moments, with respect to

$Y_{0,j}$, and obtain

$$
\begin{aligned}
E[a_1({}_\tau\mathbf{T}_j)|Y_{0,j}] &= \frac{1}{\alpha_j}Y_{0,j} + E[a_1({}_{\tau'}\mathbf{Y}_j)|Y_{0,j}] \\
&= \frac{1}{\alpha_j}Y_{0,j} + \alpha_1({}_{\tau'}Y_{1,j}|Y_{0,j}) \\
&= \frac{1}{\alpha_j}Y_{0,j} + \frac{\gamma_j}{\alpha_j}K_1, \qquad (4) \\
E[\widetilde{m}_2({}_\tau\mathbf{T}_j)|Y_{0,j}] &= E[\widetilde{m}_2({}_{\tau'}\mathbf{Y}_j)|Y_{0,j}] = \mu_2({}_{\tau'}Y_{1,j}|Y_{0,j}) \\
&= \alpha_2({}_{\tau'}Y_{1,j}|Y_{0,j}) - \alpha_1^2({}_{\tau'}Y_{1,j}|Y_{0,j}) \\
&= \frac{\gamma_j(\gamma_j+1)}{\alpha_j^2}K_2 - \frac{\gamma_j^2}{\alpha_j^2}K_1^2, \qquad (5) \\
E[\widetilde{m}_3({}_\tau\mathbf{T}_j)|Y_{0,j}] &= E[\widetilde{m}_3({}_{\tau'}\mathbf{Y}_j)|Y_{0,j}] = \mu_3({}_{\tau'}Y_{1,j}|Y_{0,j}) \\
&= \alpha_3({}_{\tau'}Y_{1,j}|Y_{0,j}) - 3\alpha_2({}_{\tau'}Y_{1,j}|Y_{0,j})\alpha_1({}_{\tau'}Y_{1,j}|Y_{0,j}) + 2\alpha_1^3({}_{\tau'}Y_{1,j}|Y_{0,j}) \\
&= \frac{\gamma_j(\gamma_j+1)(\gamma_j+2)}{\alpha_j^3}K_3 - 3\frac{\gamma_j^2(\gamma_j+1)}{\alpha_j^3}K_2K_1 + 2\frac{\gamma_j^3}{\alpha_j^3}K_1^3, \qquad (6)
\end{aligned}
$$

where the arguments of the K functions above are $(\tau', \gamma_j, \alpha_j)$. Notice that everything in the above system of equations is written in terms of raw moments, such that Lemma 1 can be utilized. Upon careful inspection, we furthermore notice that equations (4)-(6) are generalizations of equations (1)-(3). The latter set of equations are obtained when the K functions take value one, which only occurs when the truncation point is zero (i.e. when there is no truncation).

Unfortunately the above system of equations is highly unstable and cannot be solved using either iterative or numerical procedures.

3.2.2 The Simplified Case

In addition to the assumptions that $\alpha_0 = 1$ and $\tau_j = \tau$, for all j, we further assume that $\gamma_j = \gamma$ and $\alpha_j = \alpha$, for all j. This additional assumption is equivalent to assuming that lives in every pool have similar risk profiles. The level of dependence within pools, however, still varies since this depends on the value $Y_{0,j}$.

In this simplified case, we begin our estimation procedure by combining all pools. Define ${}_\tau\mathbf{T} = ({}_\tau T_{1,1}, \ldots, {}_\tau T_{N,M})'$. Due to our simplifying assumptions, the components of ${}_\tau\mathbf{T}$ are identically distributed, although not independent. This implies that the raw sample moments of ${}_\tau\mathbf{T}$ are unbiased estimators of the raw moments of ${}_\tau T_{1,1}$. Recall that $T_{i,j} \sim G(\tilde{\gamma} = \gamma_0 + \gamma, \alpha)$, and ${}_\tau T_{i,j}$

has a truncated $G(\tilde{\gamma} = \gamma_0 + \gamma, \alpha)$ distribution. Utilizing the first two raw moments, we obtain the following:

$$
\begin{aligned}
E[a_1({}_\tau\mathbf{T})] &= \alpha_1({}_\tau T_{1,1}) = \frac{\tilde{\gamma}}{\alpha} K_1(\tau, \tilde{\gamma}, \alpha), &(7)\\
E[a_2({}_\tau\mathbf{T})] &= \alpha_2({}_\tau T_{1,1}) = \frac{\tilde{\gamma}(\tilde{\gamma}+1)}{\alpha^2} K_2(\tau, \tilde{\gamma}, \alpha). &(8)
\end{aligned}
$$

Notice that we no longer condition on a single $Y_{0,j}$. This is due to the fact that ${}_\tau\mathbf{T}$ contains M different realizations from the $G(\gamma_0, \alpha_0 = 1)$ distribution, rather than one. It is, therefore, a viable option to take expectations with respect to the $Y_{0,j}$.

Equations (7) and (8) provides a two by two system of equations, but due to the presence of the K's, requires the development of a computational algorithm to provide solutions. We apply an iterative algorithm that is found to perform exceptionally well.

Algorithm 2

1. *Assume starting values for $\tilde{\gamma}$ and α, denote them $\tilde{\gamma}(1)$ and $\alpha(1)$.*

2. *Using $\tilde{\gamma}(r)$ and $\alpha(r)$, obtain estimates for the truncation adjustment coefficients K_1 and K_2. Recall that*

$$K_k(\tau, \tilde{\gamma}(r), \alpha(r)) = \frac{\overline{Ga}(\tau, \tilde{\gamma}(r) + k, \alpha(r))}{\overline{Ga}(\tau, \tilde{\gamma}(r), \alpha(r))}.$$

3. *Substitute the values of K_1 and K_2 into equations (7) and (8), rearrange, and obtain parameter estimates $\gamma(r+1)$ and $\alpha(r+1)$:*

$$
\begin{aligned}
\alpha(r+1) &= \frac{a_1({}_\tau\mathbf{T})/K_1}{a_2({}_\tau\mathbf{T})/K_2 - a_1^2({}_\tau\mathbf{T})/K_1^2},\\
\tilde{\gamma}(r+1) &= \alpha(r+1)\frac{a_1({}_\tau\mathbf{T})}{K_1},
\end{aligned}
$$

where the arguments of the K functions above are $(\tau, \tilde{\gamma}_j(r), \alpha_j(r))$ and the sample moments of ${}_\tau\mathbf{T}$ are used to estimate the theoretical moments.

4. *Return to Step 2 with $r = r + 1$ until parameter estimates are* stable.

From Algorithm 2, we obtain parameter estimate $\widehat{\alpha}$. With this estimate in hand, we return our consideration to individual pool j. We reconsider equations (4) and (5), this time armed with estimate $\widehat{\alpha}$.

$$E[a_1({}_\tau\mathbf{T}_j)|Y_{0,j}] \approx \frac{1}{\widehat{\alpha}}Y_{0,j} + \frac{\gamma}{\widehat{\alpha}}K_1(\tau', \gamma, \widehat{\alpha}), \tag{9}$$

$$E[\widetilde{m}_2({}_\tau\mathbf{T}_j)|Y_{0,j}] \approx \frac{\gamma(\gamma+1)}{\widehat{\alpha}^2}K_2(\tau', \gamma, \widehat{\alpha}) - \frac{\gamma^2}{\widehat{\alpha}^2}K_1(\tau', \gamma, \widehat{\alpha})^2. \tag{10}$$

Again, we are presented with a non-trivial system of equations. We apply the following iterative algorithm.

Algorithm 3

1. *Assume starting values for* $Y_{0,j}$ *and* γ*, denote them* $Y_{0,j}(1)$ *and* $\gamma(1)$*.*

2. *Using* $Y_{0,j}(r)$ *and* $\gamma(r)$*, obtain estimates for the truncation adjustment coefficients* K_1 *and* K_2*, where*

$$K_k\left(\tau - \frac{Y_{0,j}(r)}{\widehat{\alpha}}, \gamma(r), \widehat{\alpha}\right) = \frac{\overline{Ga}\left(\tau - \frac{Y_{0,j}(r)}{\widehat{\alpha}}, \gamma(r)+k, \widehat{\alpha}\right)}{\overline{Ga}\left(\tau - \frac{Y_{0,j}(r)}{\widehat{\alpha}}, \gamma(r), \widehat{\alpha}\right)}.$$

3. *Substitute the values of* K_1 *and* K_2 *into equations* (9) *and* (10)*, rearrange, and obtain* $Y_{0,j}(r+1)$ *and* $\gamma(r+1)$*:*

$$\gamma(r+1) = \frac{-K_2 + \sqrt{K_2^2 + 4(K_2 - K_1^2)\tilde{m}_2({}_\tau\mathbf{T}_j)\widehat{\alpha}^2}}{2(K_2 - K_1^2)},$$
$$Y_{0,j}(r+1) = a_1({}_\tau\mathbf{T}_j)\widehat{\alpha} - \gamma(r+1)K_1,$$

where the arguments of the K *functions above are* $(\tau - \frac{Y_{0,j}(r)}{\widehat{\alpha}}, \gamma(r), \widehat{\alpha})$ *and the sample moments of* ${}_\tau\mathbf{T}$ *are used to estimate the theoretical moments.*

4. *Return to Step 2 with* $r = r+1$ *until parameter estimates are* stable.

Remarks 4 (implementing Algorithm 3)

- *In Step 3, the expression under the square root can become negative; when this occurs, we set it equal to zero. The resulting estimate of* γ *is positive since* $(K_2 - K_1^2)$ *is necessarily negative.*

- *In Step 3,* $Y_{0,j}$ *can become negative. When this occurs, we set it equal to zero.*

- *As a result of the above two* corrections*, the iterative procedure can oscillate between two fixed points, with the solution typically lying in the middle. When this occurs, we take the average over these two oscillating points.*

- *For this algorithm to produce sensible results,* $Y_{0,j}(1)$ *must be* close enough *to the eventual stable value.*

As an alternative to applying Algorithm 3, one can solve (9) and (10) using a numerical method for solving nonlinear systems of equations. We make use of a strategy using different Barzilai-Borwein, see Barzilai and Borwein (1988), steplengths found in R-package BB, see Varadhan and Gilbert (2009). This procedure occasionally provides negative predictions of $Y_{0,j}$.

To complete the estimation procedure, we set

$$\widehat{\gamma} = \frac{1}{M}\sum_{j=1}^{M}\widehat{\gamma}^{(j)}, \quad \text{and} \quad \widehat{\gamma}_0 = \frac{1}{M}\sum_{j=1}^{M}\widehat{Y}_{0,j},$$

where $\widehat{\gamma}^{(j)}$ and $\widehat{Y}_{0,j}$ are the estimate of γ and predicted value of $Y_{0,j}$, respectively, obtained using Algorithm 3 on pool j. Alternatively, we have

$$\widehat{\gamma}^{(BB)} = \frac{1}{M}\sum_{j=1}^{M}\widehat{\gamma}^{(j,BB)}, \quad \text{and} \quad \widehat{\gamma}_0^{(BB)} = \frac{1}{M}\sum_{j=1}^{M}\widehat{Y}_{0,j}^{(BB)},$$

where $\widehat{\gamma}^{(j,BB)}$ and $\widehat{Y}_{0,j}^{(BB)}$ are the estimate of γ and predicted value of $Y_{0,j}$, respectively, obtained using the Barzilai-Borwein numerical procedure.

3.3 Numerical Results

In this section we study estimation performance using simulation. We investigate accuracy with and without the presence of truncation.

In Table 1, the simulation results are presented for the case with no truncation. Here, we investigate the estimation procedure for one pool, $j = 1$. It is evident that estimation is highly irratic for small N. Furthermore, the estimation appears dependent on the value of $Y_{0,1}$, where a low value of $Y_{0,1}$ produces more desirable results.

In Table 2, simulation results are presented in the face of truncated observations. Recall that under truncation, we assume each pool has the same risk profile, given by parameters γ and α.

Simulations 1 and 2 are based on only one pool. From a theoretical standpoint, it is clear that it is not appropriate to undertake parameter

Simulation	1	2	3	4	5	6
N	100	1,000	10,000	100,000	1,000,000	10,000,000
$Y_{0,1}$	7.107	1.493	4.030	3.762	4.518	4.111
$\widehat{Y}_{0,1}$	-26.258	0.038	0.152	3.948	4.214	4.169
γ_1	30.000	30.000	30.000	30.000	30.000	30.000
$\widehat{\gamma}_1$	87.727	32.166	38.939	29.718	30.792	29.847
α_1	0.500	0.500	0.500	0.500	0.500	0.500
$\widehat{\alpha}_1$	0.827	0.514	0.573	0.498	0.507	0.499

Table 1: Simulation results with no truncation present.

estimation with only one pool. Our simulation results verify this. That is, in Simulation 1, Algorithm 2 *appears* to provide a good estimate of α. However, it is evident from the results of Simulation 2, that this is due to *chance*. In Simulation 2, we have 100 times more observations than in Simulation 1, but the estimate of α is much worse. Hence, as previously stipulated, it is not prudent to apply Algorithm 2 on one pool. From Simulation 2, it is also evident that without a good estimate of α, it is very difficult to obtain a reliable estimate of γ and prediction of Y_0 (or an estimate of γ_0 when $M > 1$).

In Simulations 4, 5, and 6, estimates of α are near exact, and the Barzilai-Borwein estimates of γ and γ_0 equally impressive. The reasons for the shortcomings of Algorithm 3 are found in Remarks 4.

Simulation	1	2	3	4	5	6
N	1,000	100,000	10,000	1,000	1,000	10,000
M	1	1	50	1,000	10,000	1,000
τ	60	60	60	60	60	60
α	0.500	0.500	0.500	0.500	0.500	0.500
$\widehat{\alpha}$	0.522	0.703	0.524	0.497	0.499	0.498
γ	30.000	30.000	30.000	30.000	30.000	30.000
$\widehat{\gamma}$	26.366	59.966	32.407	28.692	28.934	29.044
$\widehat{\gamma}^{(BB)}$	32.991	59.966	33.415	29.875	30.032	29.718
Y_0 / γ_0	2.191	12.680	5.000	5.000	5.000	5.000
$\widehat{Y}_0$ / $\widehat{\gamma}_0$	7.741	0.000	4.421	6.415	6.241	5.992
$\widehat{Y}_0$ / $\widehat{\gamma}_0$ $^{(BB)}$	-0.116	-0.001	3.179	4.949	4.882	5.173

Table 2: Simulation results with truncation present.

4 Applications

In this section we provide important results for actuarial analysis as a consequence of modelling lifetimes using the multivariate gamma distribution.

4.1 Maximum Survival Time Probability

An important characteristic in survival theory is the maximum survival time in pool j, $T_{(N),j} = \max(T_{1,j}, \ldots, T_{N,j})$. It is important to know the probability that the maximum survival time exceeds t,

$$P(T_{(N),j} > t) = 1 - \int_0^\infty P(T_{(N),j} \leq t | Y_{0,j} = y_{0,j}) g(y_{0,j}, \gamma_0, \alpha_0) dy_{0,j},$$

where $g(y, \gamma, \alpha)$ is the density of $G(\gamma, \alpha)$. For a given $Y_{0,j} = y_{0,j}$, the $T_{1,j}, \ldots, T_{N,j}$ are independent and identical distributed, as a result we have

$$\begin{aligned} P(T_{(N),j} > t) &= 1 - \int_0^\infty P(Y_{1,j} \leq t - \frac{\alpha_0}{\alpha_j} y_{0,j})^N g(y_{0,j}, \gamma_0, \alpha_0) dy_{0,j} \\ &= 1 - \int_0^\infty Ga(t - \frac{\alpha_0}{\alpha_j} y_{0,j}, \gamma_j, \alpha_j)^N g(y_{0,j}, \gamma_0, \alpha_0) dy_{0,j}, \end{aligned}$$

where $Ga(y, \gamma, \alpha)$ is the distribution function of $G(\gamma, \alpha)$.

Maximum Survival for Truncated Observations

In the case of truncated observations, we analogously define the maximum survival time in pool j, ${}_\tau T_{(N),j} = \max({}_\tau T_{1,j}, \ldots, {}_\tau T_{N,j})$. We similarly obtain the probability that it exceeds $\tau + t$ as follows:

$$\begin{aligned} P({}_\tau T_{(N),j} > \tau + t) &= 1 - \int_0^\infty P({}_\tau T_{(N),j} \leq \tau + t | Y_{0,j} = y_{0,j}) g(y_{0,j}, \gamma_0, \alpha_0) dy_{0,j} \\ &= 1 - \int_0^\infty \frac{P(\tau < T_{1,j} \leq \tau + t | Y_{0,j} = y_{0,j})^N}{P(T_{1,j} > \tau | Y_{0,j} = y_{0,j})^N} g(y_{0,j}, \gamma_0, \alpha_0) dy_{0,j} \\ &= 1 - \int_0^\infty \frac{(Ga(\tau + t - \frac{\alpha_0}{\alpha_j} y_{0,j}, \gamma_j, \alpha_j) - Ga(\tau - \frac{\alpha_0}{\alpha_j} y_{0,j}, \gamma_j, \alpha_j))^N}{\overline{Ga}(\tau + t - \frac{\alpha_0}{\alpha_j} y_{0,j}, \gamma_j, \alpha_j)^N} g(y_{0,j}, \gamma_0, \alpha_0) dy_{0,j}, \end{aligned}$$

where $\overline{Ga}(y, \gamma, \alpha)$ is the survival function of $G(\gamma, \alpha)$.

4.2 The Distribution of Survival

Let $S_{t,j}$ denote the number of individuals in pool j alive at time t, hence,

$$S_{t,j} = \sum_{i=1}^N I_{\{T_{i,j} > t\}},$$

where $I_{\{T_{i,j}>t\}} = 1$ if $T_{i,j} > t$, and zero, otherwise. We are interested in the distribution of $S_{t,j}$. Define $\mathbf{T}_j = (T_{1,j}, T_{2,j}, \ldots, T_{N,j})'$ and $\mathbf{t}$ to be an N dimensional vector where every component is equal to t.

$$\begin{aligned}
P(S_{t,j} = 0) &= P(\mathbf{T}_j \leq \mathbf{t}) = P(T_{(N),j} \leq t), \\
P(S_{t,j} = n) &= \binom{N}{n} P(T_{1,j} > t, \ldots, T_{n,j} > t, T_{n+1,j} \leq t, \ldots, T_{N,j} \leq t), \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad n \in \{1, \ldots, N-1\}, \\
P(S_{t,j} = N) &= P(\mathbf{T}_j > \mathbf{t}) = P(T_{(1),j} > t),
\end{aligned}$$

where $T_{(1),j} = \min(T_{1,j}, ..., T_{N,j})$. We condition on $Y_{0,j} = y_{0,j}$ and obtain, for $i \in \{0, \ldots, N\}$,

$$\begin{aligned}
&P(S_{t,j} = n) \\
&= \int_0^\infty \binom{N}{n} P(T_{1,j} > t | Y_{0,j} = y_{0,j})^n P(T_{1,j} \leq t | Y_{0,j} = y_{0,j})^{N-n} g(y_{0,j}, \gamma_0, \alpha_0) dy_{0,j} \\
&= \int_0^\infty \binom{N}{n} P(Y_{1,j} > t - \frac{\alpha_0}{\alpha_j} y_{0,j})^n P(Y_{1,j} \leq t - \frac{\alpha_0}{\alpha_j} y_{0,j})^{N-n} g(y_{0,j}, \gamma_0, \alpha_0) dy_{0,j} \\
&= \int_0^\infty \binom{N}{n} \overline{Ga}(t - \frac{\alpha_0}{\alpha_j} y_{0,j}; \gamma_j, \alpha_j)^n Ga(t - \frac{\alpha_0}{\alpha_j} y_{0,j}; \gamma_j, \alpha_j)^{N-n} g(y_{0,j}, \gamma_0, \alpha_0) dy_{0,j}.
\end{aligned}$$

It is clear that conditional on $Y_{0,j} = y_{0,j}$, $S_{t,j}$ follows a binomial distribution where the probability of success is given by $\overline{Ga}(t - \frac{\alpha_0}{\alpha_j} y_{0,j}; \gamma_j, \alpha_j)$.

The Distribution of Survival for Truncated Lifetimes

Given truncation point τ, let ${}_\tau S_{t,j}$ denote the number of individuals in pool j alive at time t, hence,

$${}_\tau S_{t,j} = \sum_{i=1}^{N} I_{\{{}_\tau T_{i,j} > \tau + t\}}.$$

Note that the members of this pool are all aged τ at inception (i.e. when $t = 0$). As a result, we obtain that

$$\begin{aligned}
&P({}_\tau S_{t,j} = n) \\
&= \int_0^\infty \binom{N}{n} \frac{P(T_{1,j} > \tau + t | Y_{0,j} = y_{0,j})^n P(\tau < T_{1,j} \leq \tau + t | Y_{0,j} = y_{0,j})^{N-n}}{P(T_{1,j} > \tau | Y_{0,j} = y_{0,j})^N} g(y_{0,j}, \gamma_0, \alpha_0) dy_{0,j}.
\end{aligned}$$

The distribution of the number of individuals alive is critical in valuing a bulk annuity, which is an annuity on a portfolio of individuals. We investigate this next.

4.3 Annuity Valuation: Actuarial Present Value

Let $A_{t,j}$ denote the value of a bulk annuity sold to members in pool j at time t. The annuity pays \$1 at the *end* of each year to the surviving members of the pool. We are interested in the actuarial present value of $A_{0,j} \equiv A_j$, the price of such an annuity at inception.

$$A_j = \sum_{t=1}^{\infty} S_{t,j} v^t,$$

where $v = e^{-\delta}$, the discount factor with constant force of interest δ. Alternatively, A_j could be represented using the individual lifetimes:

$$A_j = \sum_{i=1}^{N} \sum_{t=1}^{\infty} I_{\{T_{i,j} > t\}} v^t.$$

Annuity Valuation for Truncated Lifetimes

Let ${}_{\tau}A_{t,j}$ be the truncated analog of $A_{t,j}$ and define ${}_{\tau}A_j \equiv {}_{\tau}A_{0,j}$. We obtain

$${}_{\tau}A_j = \sum_{t=1}^{\infty} {}_{\tau}S_{t,j} v^t,$$

alternatively,

$${}_{\tau}A_j = \sum_{i=1}^{N} \sum_{t=1}^{\infty} I_{\{{}_{\tau}T_{i,j} > \tau + t\}} v^t.$$

The actuarial present value is obtained by taking the expectation of ${}_{\tau}A_j$. We calculate this value under two different assumptions. The first considers the dependence between the lifetimes, which means expectation is taken with respect to the multivariate gamma distribution outlined above. The second treats the lifetimes as independent, which means expectation is taken with respect to a univariate gamma distribution. In both cases, the expected value is obtained using numerical methods. We compare numerical integration with simulation.

4.4 Numerical Results

Table 3 presents both numerical integration and simulation results for two sets of parameter values and various portfolio sizes. The annuity valuation is based on each surviving individual receiving \$1 at the end of each year. The results show that the dependence induced by the multivariate gamma (MVG) distribution does not play a significant role in the actuarial present value of bulk annuities. In contrast, the uncertainty, as quantified by the standard deviation, in the annuity valuation is severely affected by the presence of this dependence. For the independent case the standard deviation scales with the square root of the portfolio size. For the dependent case, the scaling factor is much larger and increases with portfolio size significantly more than in the independent case.

Parameter values							
N		1	10	100	1	10	100
δ		2%	2%	2%	2%	2%	2%
α		0.5	0.5	0.5	0.5	0.5	0.5
α_0		1.0	1.0	1.0	1.0	1.0	1.0
γ		35.0	35.0	35.0	30.0	30.0	30.0
γ_0		5.0	5.0	5.0	10.0	10.0	10.0
τ		60.0	60.0	60.0	60.0	60.0	60.0
Theoretical results obtained with numerical integration							
$E[{}_\tau A_j]$	MVG	15.81	158.11	1,571.99	15.73	157.24	1,482.70
	Ind.	15.81	158.11	1,581.07	15.73	157.32	1,573.20
Simulation results							
M (000's)		1,000	10	10	10	10	10
Mean	MVG	15.81	158.48	1,581.70	15.75	157.40	1,572.08
	Ind.	15.81	158.86	1,588.39	15.75	158.95	1,589.29
Standard	MVG	7.46	33.00	253.21	7.51	41.03	356.22
Deviation	Ind.	7.46	23.42	74.59	7.51	23.54	75.00

Table 3: Annuity valuation.

Remark 5 *The numerical integration we applied (using statistical computing software package R) tends to undervalue the bulk annuity. In various scenarios, it is a marked undervaluation; see e.g. the last column of Table 3. The source of this issue relates to inaccuracies when numerically calculating* $P({}_\tau S_{t,j} = n)$ *for large* n.

The model facilitates the assessment of capital requirements for an insurer allowing for dependence. Capital requirements based on the standard deviation of the annuity values will be significantly higher than for the case of independence, an assumption that usually underlies valuation using standard actuarial techniques.

5 Fitting Norwegian Population Data

We show how the model can be calibrated by fitting to Norwegian population data obtained from the Human Mortality Database (HMD), Human Mortality Database (2011). The HMD publishes aggregate mortality rates and exposures rather than individual lifetimes. Therefore, we transform the given mortality rates and exposure levels into *crude* lifetime data. Although there are factors – such as migration – that must needs be addressed if the goal is to obtain as accurate a picture of individual lifetimes as possible, we abstract from these details to illustrate the broad results.

We make use of cohort data from birth years 1846-1898. These were the only *complete* cohorts in the data at the time of acquisition. We consider each of these 53 cohorts as a group or pool. We assume that all cohorts share the same risk characteristics but with potentially varying levels of dependence. That is, we incorporate the assumptions associated with the *simplified case* discussed in Section 3.2.2. We have a total of $2,399,610$ deaths; by applying a truncation point of 60, this number decreases to $1,234,957$.

We begin by applying Algorithm 2. Using this algorithm, we fit the entire population in order to obtain an estimate of α. We obtain the estimate $\widehat{\alpha} = 0.87$. The plot of the histogram versus the fitted gamma density function is given in Figure 1.

With this estimate of α, we can apply either Algorithm 3 or the Barzilai-Borwein procedure in order to obtain an estimate of γ. In this example, we use the Barzilai-Borwein procedure, consequently we obtain $\widehat{\gamma} = 67.52$. The plot of the histogram versus the fitted gamma density function for the cohort born in 1885 is given in Figure 2. For this particular cohort, the dependence level is predicted to be $\widehat{Y}_{0,1885} = 6.26$.

5.1 Adjusting the Data to Achieve a Better Fit

Figures 1 and 2 show that the theoretical distribution captures some of the features of the survival distribution data but is not an ideal fit. This is a common feature of many parametric survival distributions. In this section we discuss a method of improving the fit by performing a simple adjustment to the data. The adjustment we consider is the following,

$$x'_{i,j} = \omega - x_{i,j},$$

where $x_{i,j}$ denotes the lifetime observation and ω is some highest attainable (theoretical) age. In our example, we set $\omega = 110$, but we note that ω could be set greater than the maximum observed lifetime. Ideally, ω would be estimated from the data but for illustrative purposes we take it as given

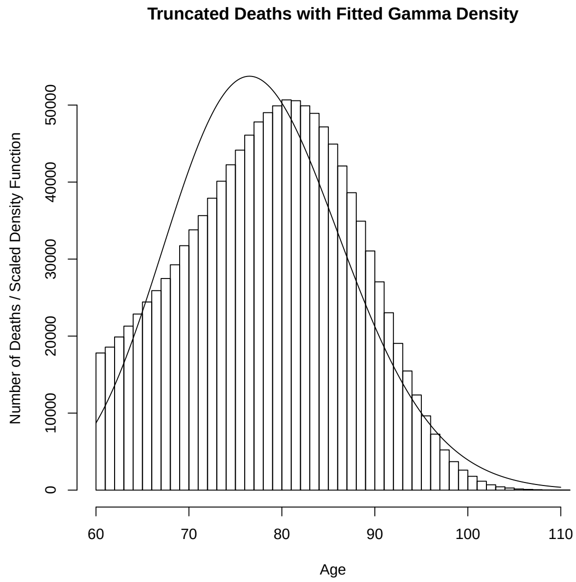

Figure 1: Histogram of population deaths versus fitted gamma density function.

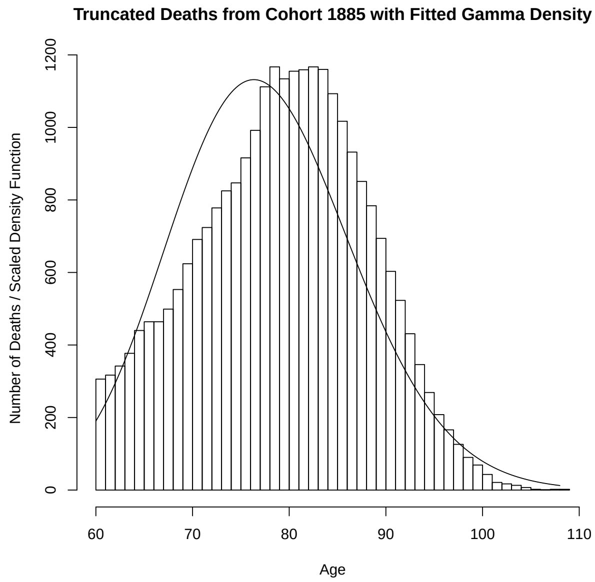

Figure 2: Histogram of deaths from those born in 1885 versus fitted gamma density function.

We operate under the assumption that the transformed data follows a multivariate gamma distribution. Consequently the theory largely stays the same. However, there is one critical difference, where the original data is truncated from the left, the transformed data is truncated from the right. This requires a new fitting procedure to be developed in order to estimate parameter values. To demonstrate the concept, we omit truncation considerations and fit a straight forward un-truncated multivariate gamma distribution.

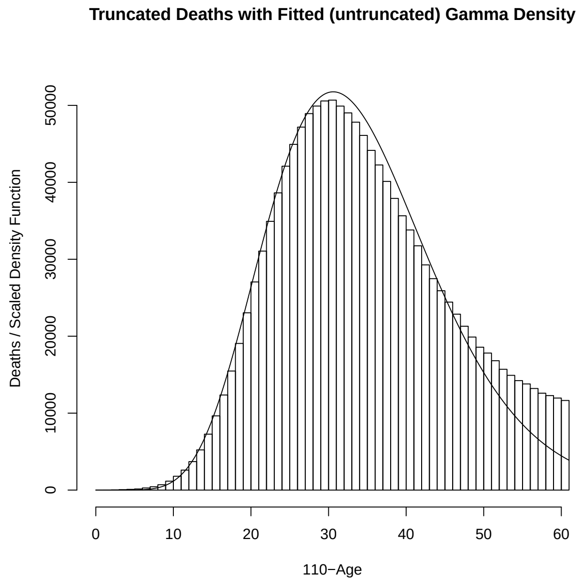

Figure 3: Histogram of deaths versus fitted (untruncated) gamma density function.

We fit the transformed data with a right truncation point of 60, which corresponds to a left truncation point of 50 in the original data. The *extension* of the truncation point ensures a slightly better fit. When fitting the entire population, we make use of Algorithm 2, which reduces to equations (7) and (8) with adjustment coefficients set to unity. When fitting the cohort from 1885, we make use of Algorithm 3 with the given value of $\widehat{\alpha}$, which reduces to equations (9) and (10) with adjustment coefficients set to unity.

Figure 3 shows the fit from the entire population; the relevant estimated parameter is $\widehat{\alpha} = 0.28$. Figure 4 shows the fit from those born in 1885; the additional estimated parameter is $\widehat{\gamma} = 9.24$ and the predicted translation point is $\widehat{Y}_{0,1885} = 0.09$. The fact that Y_0 is small could indicate a low level of dependence, but could also imply that ω has been set too low.

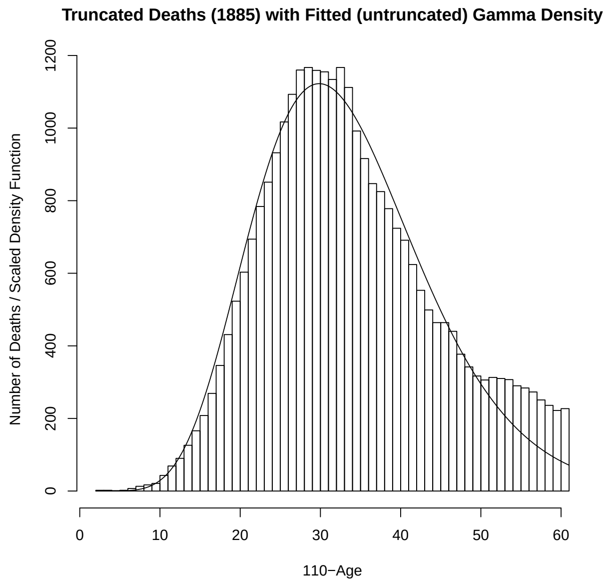

Figure 4: Histogram of deaths from those born in 1885 versus fitted (untruncated) gamma density function.

The fit from the transformed data looks promising in both scenarios, that is, in fitting the entire population and fitting one cohort. These fitted distributions improve once right truncation is accounted for in the parameter estimation procedure. This is a topic for future research.

6 Conclusion

The aim of this paper is to assess the potential impact of lifetime dependence within a pool of lives on annuity valuation and risk management. Underlying our assumption of dependent lifetimes is the presence of systematic changes in mortality, especially on a cohort level. We postulate the use of the multivariate gamma distribution for two reasons. First, due to the fact that the gamma distribution has previously been applied to model survival times. And second, due to the manner in which the multivariate gamma distribution induces dependence. That is, it exactly serves our underlying assumption and allows for mortality to be categorized into systematic and idiosyncratic components.

Parameter estimation is a practical impediment to the use of the multivariate gamma distribution. In our application, data is necessarily truncated.

We resolve the issue of parameter estimation in the presence of truncation, which we consider to be the main theoretical contribution. We apply the model to assess the impact of dependence and find that the uncertainty is severely impacted. This has serious consequences to the area of annuity pricing and risk management, which we consider to be the main practical contribution.

Finally, we fit Norwegian population data and recognize the shortcomings of our model. However, as a result of this investigative research, we immediately identify an adjustment that could lead to a much improved fit. We intend to explore this improvement in further research. Other areas where we look to improve our model include generalizing the underlying distribution and carefully refining our parameter estimation approach.

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