ARC Centre of Excellence in Population Ageing Research

Working Paper 2019/7

Portfolio Insurance Strategies for a Target Annuitization Fund

Mengyi Xu, Michael Sherris and Adam W. Shao

Portfolio Insurance Strategies for a Target Annuitization Fund

Mengyi Xu*[1,2], Michael Sherris[1,2], and Adam W. Shao[1,3]

[1]*ARC Centre of Excellence in Population Ageing Research (CEPAR), UNSW Sydney, Australia*
[2]*School of Risk and Actuarial Studies, UNSW Business School, Sydney, Australia*
[3]*Pacific Life Re, Sydney, Australia*

11th July 2019

Abstract

The transition from defined benefit to defined contribution (DC) pension schemes has increased the interest in target annuitization funds that aim to fund a minimum level of retirement income. Prior literature has studied the optimal investment strategies for DC funds that provide minimum guarantees, but far less attention has been given to portfolio insurance strategies, especially for target annuitization funds. We evaluate the performance of option-based and constant proportion portfolio insurance strategies for a DC fund that targets a minimum level of inflation-protected annuity income at retirement. We show how the portfolio allocation to an equity fund varies depending on the member's age upon joining the fund, displaying a downward trend through time for members joining the fund before ages in the mid-30s. We demonstrate how both portfolio insurance strategies provide strong protection against downside equity risk in financing a minimum level of retirement income. The option-based strategy often leads to higher accumulated savings at retirement and is also shown to provide a more robust level of protection when equity markets are more volatile and when contributions to the pension fund are lower.

Keywords: portfolio insurance strategies, defined contribution, pension risk management, target annuitization fund

JEL classification: G11, G22, G23, D14, D15, C63

*[Corresponding author]. Email: m.xu@unsw.edu.au; Postal address: CEPAR, 223 Anzac Parade, Kensington NSW 2033, Australia; Phone: +61-2-9385 7008.

1 Introduction

Occupational pension plans play an important role in the multi-pillar framework for pension systems of the World Bank (Holzmann et al., 2008). With employment-based pension shifting from defined benefit (DB) to defined contribution (DC) plans, there has been increased interest in investment strategies for these DC funds. Unlike a DB plan, a DC plan does not guarantee a life-long income stream, resulting in fund members bearing investment and longevity risk. While in theory the accumulated savings in the DC plan can be converted into a life annuity at retirement, in practice voluntary annuitization is virtually non-existent worldwide (see Brown, 2009, for a review). DC fund managers aim to maximise retirement fund values given a level of risk, without directly considering the link between the pre-retirement accumulation and post-retirement income needs.

This shift to current DC plans has led to considerations of the need to provide sustainable income flows as an investment objective (Blake et al., 2008; Financial System Inquiry, 2014). In particular, the accumulated savings in the DC fund at retirement should aim to finance a desired level of consumption during retirement. An investment product that has this aim is the "target annuitization fund" (Impavido et al., 2012). The target is probabilistic, so that the fund manager has no obligation to guarantee the targeted annuity value and there is no guarantee liability. This is compatible with the DC nature of the plan. These investment products are attracting increasing attention as they offer a link between investment accumulation and retirement income (Impavido et al., 2012).

The optimal investment strategies for DC pension plans in which the fund manager maximises the utility from retirement savings in excess of minimum guarantees has been considered (see e.g. Boulier et al., 2001; Cairns et al., 2006). In the target annuitization fund the guarantee is usually expressed in terms of a life annuity to be bought at retirement. Extensions have also been considered, for example, additional sources of risks (see e.g. Battocchio and Menoncin, 2004; Han and Hung, 2012), alternative asset price dynamics (see e.g. Guan and Liang, 2014), and different utility functions (see e.g. Blake et al., 2013; Blake et al., 2014). Another popular DC pension investment strategy is lifecycle investment, which involves a switch from mostly risky assets when plan participants are young, to more safe assets as they approach retirement, through a predetermined glide path (see Viceira, 2009, for a review). These strategies provide some protection against equity market downturn closer to retirement, although a minimum guarantee is not explicitly embedded, differing from target annuitization strategies.

In contrast, portfolio insurance strategies aim to limit downside risk and to participate on the upside. It is relevant for pension fund managers who provide minimum guarantees (Leland, 1980) and is optimal for investors with particular risk preferences (see e.g. Black and Perold, 1992; Bernard and Kwak, 2016). These strategies have potential application to target annuitization funds. Portfolio insurance strategies have been applied to DC pension plans. For example, Blake et al. (2001) compare a number of asset allocation strategies that include a constant proportion portfolio insurance (CPPI) strategy against a DB benchmark. Pézier and Scheller (2011) apply a CPPI strategy for performance sharing rules between pension fund sponsors and fund members. As well as CPPI, option-based portfolio insurance (OBPI) is a portfolio insurance strategy relevant to funds providing guarantees. A number of studies have compared OBPI and CPPI, although the results are inconclusive as to which strategy is superior (see Pézier and Scheller, 2013, for an overview).

We develop and apply both OBPI and CPPI strategies for a DC pension fund where the fund aims to provide an amount at retirement sufficient to provide an annuity-based target with a high level of confidence. We compare the two strategies by evaluating the distributions of accumulated wealth at retirement, and in particular the performance of downside risk protection under differing assumptions. We include the impact of future contributions and consider a range of entry ages into the fund.

Using simulations we show how the portfolio weights in equity, bonds and cash vary with the member's age when joining the fund. The weights in the equity fund show a downward (upward) trend for members joining the fund before (after) mid-30s. This difference mainly reflects the amount of future contributions for older aged cohorts, which are equivalent to an investment in safe assets. The portfolio weights are volatile, reflecting the impact of the volatility of the equity fund on the portfolio insurance strategies.

The constant proportion portfolio strategy provides better downside risk protection, but the degree of downside protection is sensitive to the equity market volatility and the level of contributions to the fund. We show numerically that the option-based strategy often leads to a higher average portfolio value at retirement, and its ability to ensure a minimum level of accumulated wealth is more robust.

The rest of the paper is organised as follows. Section 2 introduces the model framework. Section 3 covers the asset allocation strategies for the portfolio insurance strategies including theoretical results underlying the numerical simulations. Section 4 outlines the assumptions used in the numerical implementation. Results comparing the portfolio insurance strategies using simulations for a range of entry ages and assumptions are presented in Section 5. Section 6 concludes.

2 Financial market and fund dynamics

This section introduces the investment assets and their dynamics and the dynamics of the fund value. We develop a theoretical model in continuous time in real, or inflation-adjusted, terms and later apply the model in a discrete-time simulation study. In our analysis, to avoid complexity, we do not incorporate mortality or other exits from the fund.

2.1 The financial market

To develop the theoretical basis for our analysis, we consider a complete and frictionless financial market. The market is continuously open, has no transaction costs, borrowing constraint, taxes or margin requirements, and is assumed to preclude any arbitrage opportunities. There are three securities available for investment in the market. A money-market account or cash fund, a zero coupon bond which is traded as a constant maturity bond fund and a risky equity asset traded as an equity fund with dividends reinvested.

Risks are governed by two independent Brownian motions $Z_r(t)$ and $Z_S(t)$, where $t \geq 0$, defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Ω denotes the sample space. The filtration $\mathcal{F} = \{\mathcal{F}_t\}_{t\geq 0}$ represents the information structure generated by the Brownian motions, and $\mathbb{P}$ denotes the real world probability measure. The independence assumption for $Z_r(t)$ and $Z_S(t)$ implies no loss of generality since correlated Wiener processes can be created from uncorrelated ones via a Cholesky decomposition of the correlation matrix.

The fund manager can invest in three assets as follows. All variables are expressed in real terms.

1. A cash fund whose price M_t evolves according to

$$\frac{\mathrm{d}M_t}{M_t} = r_t \mathrm{d}t,$$

where r_t is the real interest rate, along with the discount process defined as

$$D(t) = e^{-\int_0^t r_u \mathrm{d}u} = \frac{1}{M_t}.$$

The real interest rate r_t follows an Ornstein-Uhlenbeck process

$$\mathrm{d}r_t = \kappa(\overline{r} - r_t)\mathrm{d}t + \sigma_r \mathrm{d}Z_r(t), \tag{1}$$

where $\overline{r}$ describes the long-run mean of the real interest rate, κ describes the degree of mean reversion, and σ_r is the real interest rate volatility.

2. An inflation-indexed zero-coupon bond that matures at a specified time of retirement T ($T > t$) and, given a constant market price of real interest rate risk, λ_r, has price given by

$$P(t,T) = \alpha(t,T)e^{-\beta(t,T)r_t},$$

where

$$\alpha(t,T) = \exp\left\{ \left(\overline{r} - \frac{\sigma_r\lambda_r}{\kappa} - \frac{\sigma_r^2}{2\kappa^2}\right)\left[\beta(t,T) - T + t\right] - \frac{\sigma_r^2}{4\kappa}\left(\beta(t,T)\right)^2 \right\},$$
$$\beta(t,T) = \frac{1}{\kappa}\left(1 - e^{-\kappa(T-t)}\right).$$

The stochastic process for the zero-coupon bond price, under the $\mathbb{P}$ measure, is therefore given by

$$\frac{\mathrm{d}P(t,T)}{P(t,T)} = \left[r_t - \beta(t,T)\sigma_r\lambda_r\right]\mathrm{d}t - \sigma_r\beta(t,T)\mathrm{d}Z_r(t). \tag{2}$$

Because it is unrealistic to assume the existence of zero-coupon bonds with any maturity (Boulier et al., 2001), investment is in a constant-maturity bond fund, with a constant maturity ($\overline{T}$), whose price $P_t^{\overline{T}}$ evolves according to

$$\frac{\mathrm{d}P_t^{\overline{T}}}{P_t^{\overline{T}}} = (r_t - \sigma_{\overline{T}}\lambda_r)\,\mathrm{d}t - \sigma_{\overline{T}}\mathrm{d}Z_r(t),$$

where $\overline{T}$ is the constant maturity of the bond, and

$$\sigma_{\overline{T}} = \frac{1 - e^{-\kappa\overline{T}}}{\kappa}\sigma_r.$$

The original zero-coupon bond price dynamics (2) can be obtained through a linear combination of the cash fund and the bond fund

$$\frac{\mathrm{d}P(t,T)}{P(t,T)} = \left(1 - \frac{\sigma_r\beta(t,T)}{\sigma_{\overline{T}}}\right)\frac{\mathrm{d}M_t}{M_t} + \frac{\sigma_r\beta(t,T)}{\sigma_{\overline{T}}}\frac{\mathrm{d}P_t^{\overline{T}}}{P_t^{\overline{T}}}. \tag{3}$$

3. An equity fund (with dividend reinvested) whose price S_t satisfies the following stochastic differential equation

$$\frac{\mathrm{d}S_t}{S_t} = (r_t + \sigma_S\lambda_S + \sigma_{Sr}\lambda_r)\,\mathrm{d}t + \sigma_S\mathrm{d}Z_S(t) + \sigma_{Sr}\mathrm{d}Z_r(t), \tag{4}$$

where σ_S is the equity fund's own volatility, σ_{Sr} equity fund volatility associated with interest rate, and λ_S the market price of equity fund risk. Using the following notation

$$\boldsymbol{\sigma} = \begin{pmatrix} \sigma_S \\ \sigma_{Sr} \end{pmatrix}, \quad \boldsymbol{\lambda} = \begin{pmatrix} \lambda_S \\ \lambda_r \end{pmatrix}, \quad \boldsymbol{Z}(t) = \begin{pmatrix} Z_S(t) \\ Z_r(t) \end{pmatrix},$$

the diffusion process of S_t can also be written as

$$\frac{\mathrm{d}S_t}{S_t} = \left(r_t + \boldsymbol{\sigma}^\top\boldsymbol{\lambda}\right)\mathrm{d}t + \boldsymbol{\sigma}^\top\mathrm{d}\boldsymbol{Z}(t), \tag{5}$$

where $^\top$ denotes the matrix transpose.

2.2 Fund value dynamics

We denote by $W_M(t)$, $W_{\overline{T}}(t)$, $W_S(t)$ the wealth invested in the cash fund, the constant-maturity bond fund, and the equity fund, respectively for a given group of individuals with the same entry age to the fund. In the DC pension fund, we also assume a real, continuous and deterministic contribution amount, $c(t)$, is contributed to the fund from entry to the retirement time.

The fund value at time t, X_t, is equal to

$$X_t = W_M(t) + W_{\overline{T}}(t) + W_S(t).$$

The fund value satisfies the following stochastic differential equation

$$\begin{aligned}\mathrm{d}X_t &= r_t X_t \mathrm{d}t + W_{\overline{T}}(t)\Big[-\sigma_{\overline{T}}\lambda_r \mathrm{d}t + \sigma_{\overline{T}}\mathrm{d}Z_r(t)\Big] \\ &\quad + W_S(t)\Big[\big(\sigma_S\lambda_S + \sigma_{Sr}\lambda_r\big)\mathrm{d}t + \sigma_S \mathrm{d}Z_S(t) + \sigma_{Sr}\mathrm{d}Z_r(t)\Big] + c(t)\mathrm{d}t.\end{aligned} \tag{6}$$

X_t is not a self-financing process in that the change of portfolio value is not entirely driven by the gains or losses of investment returns due to the continuous contribution to the fund. We will add in the present value of future contributions as an asset of the fund so that the total of the investments of the fund plus the present value of future contributions is a self-financing portfolio.

The present value of the (inflation-indexed) future contributions is denoted by L_t with value given by

$$L_t = \int_t^T c(u)P(t,u)\mathrm{d}u.$$

Using Itô's formula, L_t can be replicated with the constant-maturity bond fund and the cash fund as follows

$$W^L_{\overline{T}}(t) = \int_t^T c(u)P(t,u)\frac{\sigma_r\beta(t,u)}{\sigma_{\overline{T}}}\mathrm{d}u, \tag{7}$$

$$W^L_M(t) = L_t - W^L_{\overline{T}}(t). \tag{8}$$

The self-financing portfolio, which includes the future contributions, is denoted by $Y_t = X_t + L_t$, and satisfies

$$\begin{aligned}\mathrm{d}Y_t = r_t Y_t \mathrm{d}t + W_{\overline{T}}^{Y}(t)\Big[-\sigma_{\overline{T}}\lambda_r \mathrm{d}t + \sigma_{\overline{T}}\mathrm{d}Z_r(t)\Big] \\ + W_S^Y(t)\Big[\big(\sigma_S\lambda_S + \sigma_{Sr}\lambda_r\big)\mathrm{d}t + \sigma_S \mathrm{d}Z_S(t) + \sigma_{Sr}\mathrm{d}Z_r(t)\Big],\end{aligned} \tag{9}$$

where

$$W_S^Y(t) = W_S(t), \quad W_{\overline{T}}^Y(t) = W_{\overline{T}}(t) + W_{\overline{T}}^L(t), \quad W_M^Y(t) = W_M(t) + W_M^L(t). \tag{10}$$

Y_t satisfies the self-financing condition

$$Y_0 = X_0 + \int_0^T c(t)P(0,t)\mathrm{d}t, \quad Y_T = X_T,$$

where X_0 is the initial amount of assets in the fund.

The important aspect of this formulation is that the equity assets in the fund and in the self-financing portfolio are the same, with adjustments made to the constant maturity bond and cash holdings to incorporate the present value of future contributions.

3 Portfolio insurance strategies

To assess the portfolio insurance strategies, we derive and analyse the asset allocation strategies based on the OBPI and CPPI for the self-financing portfolio Y_t. We then derive the value of the assets in the fund using Equation (10) to adjust for the present value of fund contributions to give the wealth management processes, $W_M(t)$, $W_{\overline{T}}(t)$, and $W_S(t)$ for the target annuitization fund.

3.1 Target

The investment target is based on the present value of an inflation-indexed annuity with annual payments providing a minimum level of desirable post-retirement consumption. We denote by A_T the target annuitization level at the time of retirement, T, where A_T can be written as the value of a series of zero coupon bonds, such that

$$A_T = g\sum_{j=0}^{J} P(T, T+j),$$

where g is the annual retirement income in today's dollars, and J represents the annuity term. We do not include mortality since we assume individuals self-insure their longevity risk reflecting what happens in practice in many countries with DC plans where there is a low level of annuitization at retirement. Similarly, at any time, t, prior to retirement T, the target annuitization level is the present value of A_T, or

$$A_t = g \sum_{j=0}^{J} P(t, T+j).$$

3.2 Option-based portfolio insurance strategy

The fund is invested with the specific aim to provide the fund members with the target annuitization level as a minimum at their retirement. An investment strategy to achieve this aim is to hold a portfolio consisting of the investment in the equity fund and options to exchange the amount in the equity fund for the target annuity value at the time of retirement. This strategy is similar to a protective put that can insure against unwanted losses. Clearly such options do not trade but they provide the basis for option replication of the target annuitization level and hence the investment strategy for the fund.

The price of the exchange option and the corresponding hedging portfolio can be derived in our market set-up. We will use this to determine the investment strategy.

3.2.1 Pricing the minimum target annuitization option

We assume that a single option is based on an annuity income value of $\frac{g}{n}$, so that n options are required to finance the annuity income of g, and each option has a time 0 price of Q_0. The initial value of the self replicating portfolio, Y_0, is then given by

$$Y_0^{\mathrm{OBPI}} = n(S_0 + Q_0), \quad n > 0. \tag{11}$$

The value of the investment portfolio at retirement, for a specified group of individuals with the same entry age entering the fund at time 0, is given by

$$X_T^{\text{OBPI}} = Y_T^{\text{OBPI}} = nS_T + \left(A_T - nS_T\right)^+ = n\left[S_T + \left(\frac{g}{n}\sum_{j=0}^{J} P(T, T+j) - S_T\right)^+\right],$$

where $(\cdot)^+ = \max(0, \cdot)$.

We will specify contribution levels which, along with an initial amount of fund assets contributed at time 0 will fund the target annuitization level. There will be a minimum level of initial assets contributed to meet the target annuitization level. The option price at time 0 has a lower bound given by the following equation

$$\begin{aligned} Q_0 &= \widetilde{\mathbb{E}}\left[\frac{D(T)}{D(0)}\left(\frac{g}{n}\sum_{j=0}^{J} P(T, T+j) - S_T\right)^+\right] \geq \widetilde{\mathbb{E}}\left[\frac{D(T)}{D(0)}\left(\frac{g}{n}\sum_{j=0}^{J} P(T, T+j) - S_T\right)\right] \\ &= \frac{g}{n}\sum_{j=0}^{J} P(0, T+j) - S_0 = \frac{g}{n}A_0 - S_0, \end{aligned}$$

where $\widetilde{\mathbb{E}}$ is the expectation operator under the risk neutral probability measure $\widetilde{\mathbb{P}}$ with the cash fund M as numéraire. From this we see that the initial amount of assets, contributed by the fund members entering at time 0, will need to be such that $Y_0 \geq gA_0$ in order to provide sufficient funds to finance the minimum target benefit.

At any time t prior to time T, the value of the portfolio is given by

$$Y_t^{\text{OBPI}} = n\left(S_t + Q_t\right),$$

where Q_t is the value of a single option at time t. Using the risk-neutral pricing formula, the value of the option at time t is given by

$$Q_t = \widetilde{\mathbb{E}}_t\left[\frac{D(T)}{D(t)}S_T\left(\frac{g}{n}\sum_{j=0}^{J}\frac{P(T, T+j)}{S_T} - 1\right)^+\right].$$

We use the change-of-numéraire technique (Geman et al., 1995), changing the numéraire from cash fund M to equity fund S, to find the option price, Q_t. Denoting the risk neutral measure for the equity fund numéraire by $\widetilde{\mathbb{P}}^{(S)}$, the Radon-Nikodym derivative defining the measure $\widetilde{\mathbb{P}}^{(S)}$ is given by

$$\frac{\mathrm{d}\widetilde{\mathbb{P}}^{(S)}}{\mathrm{d}\widetilde{\mathbb{P}}} = \frac{S_T M_0}{S_0 M_T} = \exp\left(\sigma_S \widetilde{Z}_S(T) - \frac{1}{2}\sigma_S^2 T + \sigma_{Sr}\widetilde{Z}_r(T) - \frac{1}{2}\sigma_{Sr}^2 T\right).$$

The multidimensional Girsanov theorem implies that under $\widetilde{\mathbb{P}}^{(S)}$,

$$\widetilde{Z}_S^{(S)}(t) = \widetilde{Z}_S(t) - \sigma_S t \quad \text{and} \quad \widetilde{Z}_r^{(S)}(t) = \widetilde{Z}_r(t) - \sigma_{Sr} t$$

are standard Brownian motions, and that $\widetilde{Z}_S^{(S)}$ and $\widetilde{Z}_r^{(S)}$ are independent. Therefore, the option price under the risk-neutral measure $\widetilde{\mathbb{P}}^{(S)}$ is given by

$$Q_t = \frac{g}{n} S_t \widetilde{\mathbb{E}}_t^{(S)} \left[\left(\sum_{j=0}^{J} P^{(S)}(T, T+j) - \frac{n}{g}\right)^+\right], \tag{12}$$

where $\widetilde{\mathbb{E}}_t^{(S)}$ denotes the conditional expectation operator given $\mathcal{F}_t$ under the probability measure $\widetilde{\mathbb{P}}^{(S)}$, and $P^{(S)}(T, T+j)$ is the price of the zero-coupon bond denominated in S. Then $P^{(S)}(t,T)$ is a martingale under the measure $\widetilde{\mathbb{P}}^{(S)}$, with

$$\frac{\mathrm{d}P^{(S)}(t,T)}{P^{(S)}(t,T)} = -\sigma_S \mathrm{d}\widetilde{Z}_S^{(S)}(t) - (\sigma_r \beta(t,T) + \sigma_{Sr})\,\mathrm{d}\widetilde{Z}_r^{(S)}(t).$$

Since $\widetilde{Z}_S^{(S)}(t)$ and $\widetilde{Z}_r^{(S)}(t)$ are independent, we can define a new Brownian motion $\widetilde{Z}_P^{(S)}$ such that

$$\frac{\mathrm{d}P^{(S)}(t,T)}{P^{(S)}(t,T)} = \sqrt{\sigma_S^2 + (\sigma_r \beta(t,T) + \sigma_{Sr})^2}\mathrm{d}\widetilde{Z}_P^{(S)}(t) \equiv \sigma_P(t,T)\mathrm{d}\widetilde{Z}_P^{(S)}(t).$$

So for each $s \leq t$,

$$P^{(S)}(t,T) = P^{(S)}(s,T) \exp\left[\int_s^t \sigma_P(u,T)\mathrm{d}\widetilde{Z}_P^{(S)}(u) - \frac{1}{2}\int_s^t \sigma_P^2(u,T)\mathrm{d}u\right]. \tag{13}$$

Since $\sigma_P(t,T)$ is a deterministic function of time t, substituting Equation (13) into Equation (12) gives

$$Q_t = \frac{g}{n} S_t \widetilde{\mathbb{E}}_t^{(S)} \left[\left(\sum_{j=0}^{J} P^{(S)}(t, T+j) \exp\left\{ \varepsilon_{t,T} \sqrt{\int_t^T \sigma_P^2(u, T+j) \mathrm{d}u} - \frac{1}{2} \int_t^T \sigma_P^2(u, T+j) \mathrm{d}u \right\} - \frac{n}{g} \right)^+ \right], \tag{14}$$

where $\varepsilon_{t,T}$ is a random variable that follows a standard normal distribution under the measure $\widetilde{\mathbb{P}}^{(S)}$ for any $t \in [0, T]$.

Lemma 3.1. *Using the result from Jamshidian (1989), the value of the option at time t is given by*

$$Q_t = \frac{g}{n} S_t \sum_{j=0}^{J} \widetilde{\mathbb{E}}_t^{(S)} \left[\left(P^{(S)}(T, T+j) - K_j^{(S)}(t) \right)^+ \right], \tag{15}$$

where $K_j^{(S)}(t)$ is an appropriate strike price given by

$$K_j^{(S)}(t) = P^{(S)}(t, T+j) \exp\left[\varepsilon_{t,T}^* \sqrt{\int_t^T \sigma_P^2(u, T+j) \mathrm{d}u} - \frac{1}{2} \int_t^T \sigma_P^2(u, T+j) \mathrm{d}u \right], \tag{16}$$

where $\varepsilon_{t,T}^*$ satisfies the following equation

$$\sum_{j=0}^{J} P^{(S)}(t, T+j) \exp\left[\varepsilon_{t,T}^* \sqrt{\int_t^T \sigma_P^2(u, T+j) \mathrm{d}u} - \frac{1}{2} \int_t^T \sigma_P^2(u, T+j) \mathrm{d}u \right] = \frac{n}{g}. \tag{17}$$

Proof. See Appendix A.1. □

Theorem 3.2. *Based on Lemma 3.1, the value of the option at time t is given by*

$$Q_t = \frac{g}{n} S_t \sum_{j=0}^{J} \left[P^{(S)}(t, T+j) N(-d_{2,t}) - K_j^{(S)}(t) N(-d_{1,t}) \right], \tag{18}$$

where $N(\cdot)$ represents the cumulative distribution function of the standard normal distribution,

$$d_{1,t} = \frac{1}{\sqrt{\int_t^T \sigma_P^2(u, T+j) \mathrm{d}u}} \left(\ln \frac{K_j^{(S)}(t)}{P^{(S)}(t, T+j)} + \frac{1}{2} \int_t^T \sigma_P^2(u, T+j) \mathrm{d}u \right), \tag{19}$$

and

$$d_{2,t} = d_{1,t} - \sqrt{\int_t^T \sigma_P^2(u, T+j)\mathrm{d}u}. \tag{20}$$

Proof. The proof follows the derivation of the Black-Scholes formula for European call options. □

Remark 3.3. By comparing Equation (19) with Equation (16), $d_{1,t} \equiv \varepsilon^*_{t,T}$, so $d_{1,t}$ does not depend on j.

With an analytical expression for the option value under the financial model we can determine the investment strategy of the target annuitization fund.

3.2.2 Option replicating portfolio

The portfolio consists of an investment in the equity fund, and a corresponding hedging portfolio to hedge a short position in the option whose value is given by Equation (18).

To do this it is easier to find the hedging portfolio when the numéraire is the equity fund so we divide Equation (18) by S_t

$$\frac{Q_t}{S_t} = \frac{g}{n}\sum_{j=0}^{J}\left[P^{(S)}(t, T+j)N(-d_{2,t}) - K_j^{(S)}(t)N(-d_{1,t})\right]. \tag{21}$$

If we take a hold $\frac{g}{n}N(-d_{2,t})$ units of a zero-coupon bond that matures at time $T+j$ ($j = 0, \cdots, J$) and short $\frac{g}{n}\sum_{j=0}^{J} K_j^{(S)}(t)N(-d_{1,t})$ in the equity fund at each time t we see that the value of this portfolio agrees with Equation (21). Hence this is equivalent to a short position in the option.

Theorem 3.4. *The hedging portfolio in Equation* (21) *is self-financing.*

Proof. See Appendix A.2 for an outline. □

Purchasing n units of the equity fund in addition to this hedging portfolio gives the investment strategy for the fund in terms of Y_t. Therefore, for the self-financing portfolio Y_t, the wealth invested in the equity fund is

$$W_S^{Y,\ \text{OBPI}}(t) = nS_t - gS_t \sum_{j=0}^{J} K_j^{(S)}(t) N(-d_{1,t}), \tag{22}$$

and the wealth invested in the zero-coupon bond that matures at time $T + j$ is

$$W_{P(t,T+j)}^{\text{Y, OBPI}}(t) = gP(t, T + j)N(-d_{2,t}).$$

To convert our investment strategy into the constant maturity bond fund with a constant maturity $\overline{T}$, the zero-coupon bonds with these terms of maturity need to be replicated using the cash fund and the bond fund according to Equation (3).

The wealth invested in the cash fund becomes

$$W_M^{Y,\ \text{OBPI}}(t) = g \sum_{j=0}^{J} P(t, T + j)N(-d_{2,t}) \left(1 - \frac{\sigma_r \beta(t, T + j)}{\sigma_{\overline{T}}}\right). \tag{23}$$

The wealth invested in the bond fund with constant maturity $\overline{T}$ is then

$$W_{\overline{T}}^{Y,\ \text{OBPI}}(t) = g \sum_{j=0}^{J} P(t, T + j)N(-d_{2,t}) \frac{\sigma_r \beta(t, T + j)}{\sigma_{\overline{T}}}. \tag{24}$$

The self-financing portfolio, Y_t is given by: *1*) a hedging portfolio that hedges a short position in the option, *2*) n units of the equity fund. Since the target annuitization fund incorporates the value of future contributions, Y_t has to be adjusted for the value of future contributions to determine the investment strategy for the fund assets. That is, Equations (22), (23) and (24) need to be adjusted for the portfolio that represents the value of future contributions.

Doing this we obtain that, for the portfolio X_t, the wealth invested in the equity fund, bond fund, cash fund, are respectively given by

$$W_S^{\text{OBPI}}(t) = nS_t - gS_t \sum_{j=0}^{J} K_j^{(S)}(t)N(-d_{1,t}),$$
$$W_{\overline{T}}^{\text{OBPI}}(t) = g \sum_{j=0}^{J} P(t,T+j)N(-d_{2,t}) \frac{\sigma_r \beta(t,T+j)}{\sigma_{\overline{T}}} - W_{\overline{T}}^{L}(t),$$
$$W_M^{\text{OBPI}}(t) = g \sum_{j=0}^{J} P(t,T+j)N(-d_{2,t}) \left(1 - \frac{\sigma_r \beta(t,T+j)}{\sigma_{\overline{T}}}\right) - W_M^{L}(t).$$

3.3 Constant proportion portfolio insurance strategy

The CPPI strategy is implemented by determining the amount allocated to the risky asset as the product of a cushion, $\mathcal{C}_t$, and a multiplier, m (Black and Jones, 1987). The cushion is the portfolio value minus the minimum target annuitization level in our case. If the portfolio value is below this minimum value, the exposure to risky assets is zero. Hence, the exposure to the equity fund at time t is given by

$$\mathcal{E}_t = m\mathcal{C}_t = m(Y_t^{\text{CPPI}} - A_t)^+.$$

If the assets less the cushion are invested in a portfolio that replicates A_t then the dynamics of the self-financing portfolio value at time t are given by

$$\mathrm{d}Y_t^{\text{CPPI}} = (Y_t^{\text{CPPI}} - \mathcal{E}_t)\frac{\mathrm{d}A_t}{A_t} + \mathcal{E}_t \frac{\mathrm{d}S_t}{S_t}.$$

The replicating portfolio for the CPPI strategy closely tracks the minimum target annuitization level (Black and Perold, 1992) and is constructed by holding g units of a zero-coupon bond that matures at time $T+j$ ($j = 0, \cdots, J$). Since the bond fund has a constant term to maturity, the zero-coupon bonds with different terms of maturity are replicated using the cash fund and the bond fund.

In summary, for the self-financing portfolio (Y_t), the wealth invested in the cash fund is given by

$$W_M^{Y,\,\text{CPPI}}(t) = \frac{g}{A_t}\left(Y_t^{\text{CPPI}} - \mathcal{E}_t\right) \sum_{j=0}^{J} P(t,T+j) \left(1 - \frac{\sigma_r \beta(t,T+j)}{\sigma_{\overline{T}}}\right), \tag{25}$$

and the wealth invested in the constant-maturity bond fund is given by

$$W_{\overline{T}}^{Y,\,\mathrm{CPPI}}(t) = \frac{g}{A_t}\left(Y_t^{\mathrm{CPPI}} - \mathcal{E}_t\right)\sum_{j=0}^{J} P(t, T+j)\frac{\sigma_r \beta(t, T+j)}{\sigma_{\overline{T}}}. \tag{26}$$

For the actual investments in the fund portfolio (X_t), the wealth invested in each asset is found using Equations (25), (26) and the relationships given in Equation (10).

4 Model implementation assumptions

In order to implement the theory in the previous section we need to make assumptions as to dynamics of the asset returns in real terms, the level of target annuitization, the ages of individuals in the fund, initial values and future contributions, and the frequency of rebalancing.

4.1 Financial market assumptions

We base our financial market assumptions, shown in Table 1, on Brennan and Xia (2002)[1]. The real interest rate at time 0 is set at 2.5% and S_0 is set at \$1,000. We use simulation with 100,000 paths for these return processes to determine the investment strategies under different investment scenarios for both OBPI and CPPI strategies.

Table 1. Parameter values for the numerical implementation of portfolio insurance strategies.

Parameter	Value
Real interest rate: $\mathrm{d}r_t = \kappa(\overline{r} - r_t)\mathrm{d}t + \sigma_r \mathrm{d}Z_r(t)$	
κ	0.631
$\overline{r}$	0.012
σ_r	0.026
Bond fund return process: $\dfrac{\mathrm{d}P_t^{\overline{T}}}{P_t^{\overline{T}}} = (r_t - \sigma_{\overline{T}}\lambda_r)\mathrm{d}t - \sigma_{\overline{T}}\mathrm{d}Z_r(t)$	
$\overline{T}$	20
λ_r	−0.209
Equity fund return process: $\dfrac{\mathrm{d}S_t}{S_t} = (r_t + \sigma_S\lambda_S + \sigma_{Sr}\lambda_r)\mathrm{d}t + \sigma_S\mathrm{d}Z_S(t) + \sigma_{Sr}\mathrm{d}Z_r(t)$	
σ_S	0.157
σ_{Sr}	−0.020
λ_S	0.343

[1]We adjust the dynamics to obtain the equity's price dynamics in real terms by dividing the nominal price by the price index. Since Brennan and Xia (2002) show that the correlation coefficient between the stock price and inflation is close to zero, adjusting their parameter estimation provides a reasonable basis for our analysis.

4.2 Target annuitization level

We use a target fund balance at retirement providing a level of minimum income during retirement. There is currently no consensus on a minimum standard of retirement living (Binswanger and Schunk, 2012) so we use the retirement standard published by the Association of Superannuation Funds of Australia (ASFA) as a guide. It is estimated that a single person needs \$24,250 (\$23,754) per annum at around 65 (85) to maintain a modest lifestyle (Association of Superannuation Funds of Australia, 2017). Based on this we set the target fund balance such that it is expected to provide the member with an annuity of \$24,000 (in real term) every year for 35 years.

Since we do not incorporate mortality and assume an individual will self-insure their longevity, the 35-year horizon is selected to cover the period from ages 65 to 100, so that there is only a small probability of outliving the annuity income. Table 2 shows that the probabilities of living beyond 100 for 65-year-old individuals are mostly below 4% in major developed countries.

Table 2. The probability that a 65-year-old individual will live beyond age 100 using the latest available life tables. The year in the brackets denotes the year of the life table.

	Australia (2016)	Japan (2017)	U.K. (2016)	U.S. (2017)
Male	1.6%	1.6%	1.1%	1.8%
Female	3.4%	6.8%	2.4%	3.7%

Source: Human Mortality Database.

We consider eight different cohorts with 40, 35, $\cdots$, 10, 5 years of pre-retirement investment horizons representing ages at entry 25 to 60 assuming a retirement age of 65. Table 3 shows that initial value of the target level of annuitization, A_0, along with the mean and standard deviation of target annuitization levels at retirement at age 65. These are similar for different entry cohorts since the level of annuity income assumed is the same for individuals regardless of entry age and the term structure of interest rates is mostly flat.

4.3 Assets and contributions

We assume the portfolio is rebalanced annually, and that contributions are made at the beginning of each year from entry to retirement. The value of future contributions on an annual

Table 3. The target annuitization level at time 0, and the mean and standard deviation (Std) of the target annuitization level at retirement age 65 for different cohorts.

Age at time 0	A_0 (\$000)	A_T (\$000) Mean	Std
25	275.28	619.25	21.00
30	303.87	619.57	20.84
35	335.43	619.17	21.13
40	370.27	619.33	20.87
45	408.73	619.66	20.89
50	451.18	619.00	20.86
55	498.04	619.51	21.00
60	549.78	619.01	20.88

basis is given by

$$L_t = \sum_{u=t}^{T} c(u)P(t,u),$$

and $W_{\overline{T}}^{L}(t)$ given by (7) becomes

$$W_{\overline{T}}^{L}(t) = \sum_{u=t}^{T} c(u)P(t,u)\frac{\sigma_r \beta(t,u)}{\sigma_{\overline{T}}}.$$

The budget constraint of Y_t is given by

$$Y_{t+1} = W_S^{Y,\,\mathrm{PI}}(t)\frac{S_{t+1}}{S_t} + \sum_{j=0}^{J} gP(t+1, T+j)N(-d_{2,t}),$$

where PI represents either OBPI or CPPI.

For the OBPI strategy, the initial investment in the equity fund is determined by multiplying the number of options, n, by the equity fund price, S_0. n is obtained by solving Equations (11) and (18) simultaneously. For the CPPI strategy, the amount invested in the equity fund at time 0 equals the product of the multiplier, m, and the cushion amount, $\mathcal{C}_t$, which are both independent of S_0.

In terms of the multiplier in the CPPI strategy, we are interested in cases when $m > 1$, that is, when the payoff function is convex. When $m = 1$, CPPI reduces to buy-and-hold strategies.

We assume the contribution is made on an annual basis, and that it increases by 2.5% per annum to reflect labour productivity growth.[2] Table 4 shows the initial level of contribution assumed, $c(0)$ which, together with the annual growth rate of 2.5%, gives the present value of future contributions shown in the third column. We select contribution levels sufficient to fund the target benefit at age 25 with a reasonable increase with age at entry.

The fourth column shows the assumed buffer above the target annuitization level at time 0, which determines the initial fund value required along with the value of future contributions. The assumed buffer above the target affects the extent to which the fund can invest in the risky asset. We set the initial buffer to be equal across cohorts. We will consider the impact of a lower buffer on the investment strategies (Section 5.3).

The resulting initial fund balance is shown in the seventh column. It increases with age, which reflects an assumption that older aged individuals entering the fund will have accumulated retirement savings from earlier ages to provide this level of initial contribution into the fund.

Table 4. The assumption about initial values, including initial contribution and initial fund balance, used in the base case.

Age at time 0	$c(0)$ (\$000)	L_0 (\$000)	$Y_0 - A_0$ (\$000)	A_0 (\$000)	Y_0 (\$000)	$X_0 = Y_0 - L_0$ (\$000)	L_0/X_0
25	7	308.04	33	275.28	308.28	0.24	1,261.68
30	8	303.92	33	303.87	336.87	32.95	9.22
35	9	289.17	33	335.43	368.43	79.26	3.65
40	10	264.22	33	370.27	403.27	139.05	1.90
45	11	229.48	33	408.73	441.73	212.25	1.08
50	12	185.34	33	451.18	484.18	298.84	0.62
55	13	132.18	33	498.04	531.04	398.86	0.33
60	14	70.37	33	549.78	582.78	512.41	0.14

4.4 Option prices

Given S_0, Y_0, and r_0, we can solve for the number of options to be replicated, and the option price at time 0. Table 5 summarises the results. The value of the option declines as the investment horizon decreases, reflecting the decay in the time value of the option. As a result, the amount invested in the equity fund increases, and n increases for the shorter investment horizons.

[2]The average labour productivity growth rate in the U.S. was 2.3% from 1947 to 2007, and 2.7% from 2001 to 2007, although that from 2007 to 2016 declined to 1.1% (Bureau of Labor Statistics, U.S. Department of Labor, 2017).

Table 5. Number of options to be replicated and the value of a single option at time 0.

	Age at time 0							
	25	30	35	40	45	50	55	60
n	152.6	170.5	192.3	219.4	253.9	299.1	361.5	455.2
Q_0 ($)	1,020.01	975.92	915.85	837.90	739.82	618.57	469.10	280.14

5 Comparison of investment strategies

5.1 Portfolio weights

This section compares the investment strategies using the portfolio weights in each asset for the OBPI and CPPI strategies. For members joining the fund at age 25 (30), the pension fund balance in the first 10 (5) years is relatively low, so the portfolio weights in the equity fund tend to be large positive figures and those in the bond fund tend to be large negative figures. In addition, the portfolio weights are very sensitive to the fund balances. We therefore focus on the results in the last 30 years before retirement for these two cohorts. For the remaining cohorts, the results are shown for the whole pre-retirement period.

Figure 1 shows the average portfolio weights in the equity fund for members joining the pension fund at different ages for the portfolio insurance strategies. If the member joins the fund at a relatively young age, the proportion invested in the equity fund decreases as the fund member grows older. This pattern is similar to the lifecycle investment strategy in that the portfolio mix becomes more conservative as members get older. Unlike traditional lifecycle investment strategies where the portfolio mix is changed in a predetermined way, the portfolio insurance strategies respond dynamically to investment opportunities. This leads to a higher level of downside risk protection in a bear market and a better upside performance in a bull market compared to the standard lifecycle investment strategy.

If the member joins the fund after the mid-30s, the portfolio weights in the equity fund are lower and show an upward trend with age. The difference in portfolio weight between different cohorts reflects the older cohorts having lower amounts of future contributions. For younger cohorts, when they join the fund, the self-financing portfolio (Y_0) is dominated by the safe assets composed of future contributions (the last column of Table 4). This allows fund managers to invest heavily in the risky asset before turning 40. The safe assets gradually decline as

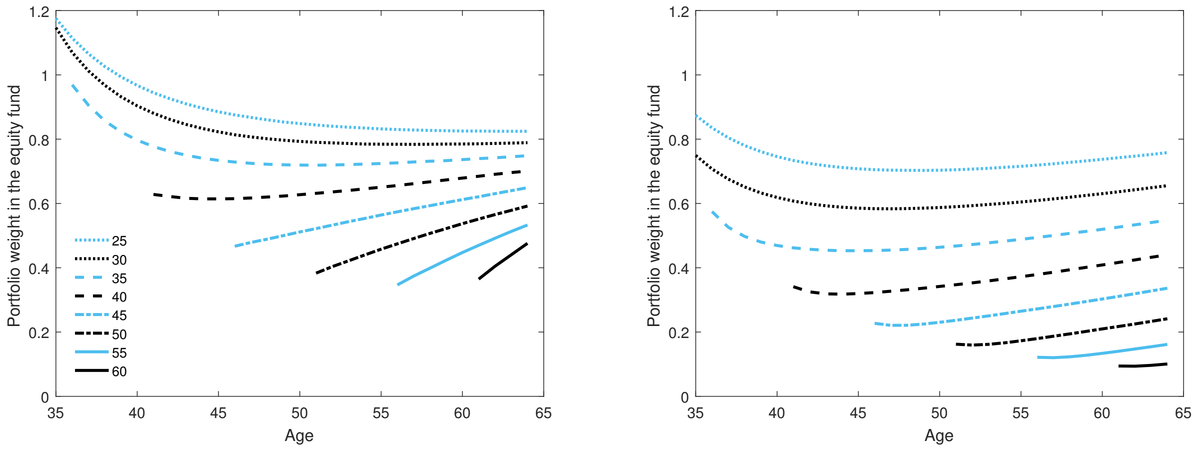


Figure 1. Average portfolio weights in the equity fund: (Left Panel) OBPI; (Right Panel) CPPI with $m = 1.6$.

members get older, and the proportion invested in the equity fund increases. Compared to the younger cohorts, older cohorts' initial fund values are significantly higher (second last column of Table 4), so the present value of future contributions remains a small proportion of the self-financing portfolio. As a result, the portfolio weights in the equity fund for the fund balance (X_t) follow the trend of the self-financing portfolio (Y_t).

The right panel of Figure 2 shows that on average, the proportion of the self-financing portfolio invested in the equity fund increases over time. This is due to the nature of the portfolio insurance strategies, where the greater the portfolio value over the target, the higher the portfolio weight in the risky asset. At time 0, Y_0 has a buffer over A_0 to allow investment in the equity fund and to allow the possibility of achieving a higher value than the target level of annuitization. The buffer will generally grow wider, since Y_t typically grows faster than A_t. This leads to a higher average proportion of wealth allocated to the equity fund at older ages.

Figure 1 also shows that the two portfolio insurance strategies generate different trends in portfolio weight for the younger cohorts after they turn 50. The weight in the equity fund slightly increases using the CPPI strategy while that of the OBPI strategy remains relatively flat. This reflects the CPPI's better downside protection than the OBPI. The downside protection can be measured using the shortfall probability and average shortfall amount. The shortfall probability is defined as the probability that the fund value is below the target annuitization level,

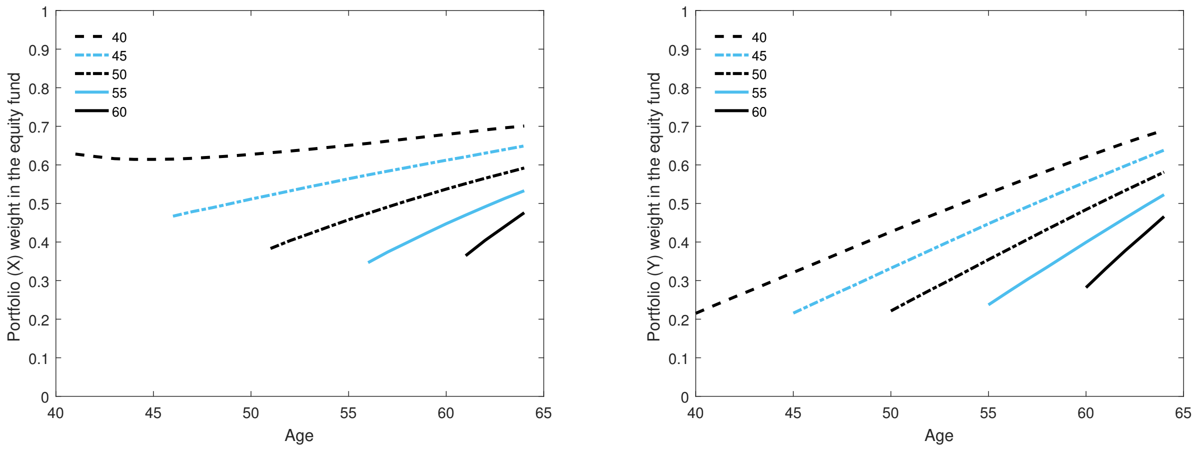


Figure 2. Average portfolio weights in the equity fund using the OBPI strategy: (Left Panel) the target annuitization fund X_t; (Right Panel) the self-financing portfolio Y_t .

i.e. $\mathbb{P}(X_t < A_t)$. The average shortfall is given by

$$\mathbb{E}\left[X_t - A_t | X_t < A_t\right],$$

where $\mathbb{E}$ is the expectation operator under the real world probability measure $\mathbb{P}$.

Figure 3 and Figure 4 show the shortfall probability and the average shortfall amount (taking the absolute dollar amount), respectively, for the youngest two cohorts. As the fund receives contributions and investment returns, both strategies give lower chances and severities of a shortfall. As the fund members approach retirement, both the shortfall probability and the average shortfall amount decrease significantly faster for the CPPI strategy.

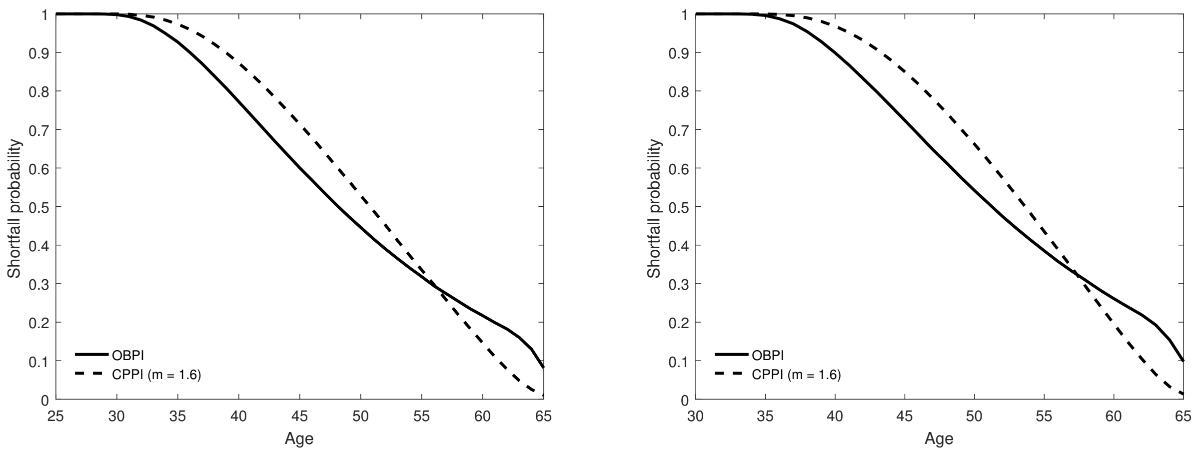


Figure 3. The shortfall probability: (Left Panel) members join the fund at age 25; (Right Panel) members join the fund at age 30. The shortfall probability is given by $\mathbb{P}(X_T < A_T)$.

The average portfolio weights in the bond fund and cash fund are shown in Figure 5 and

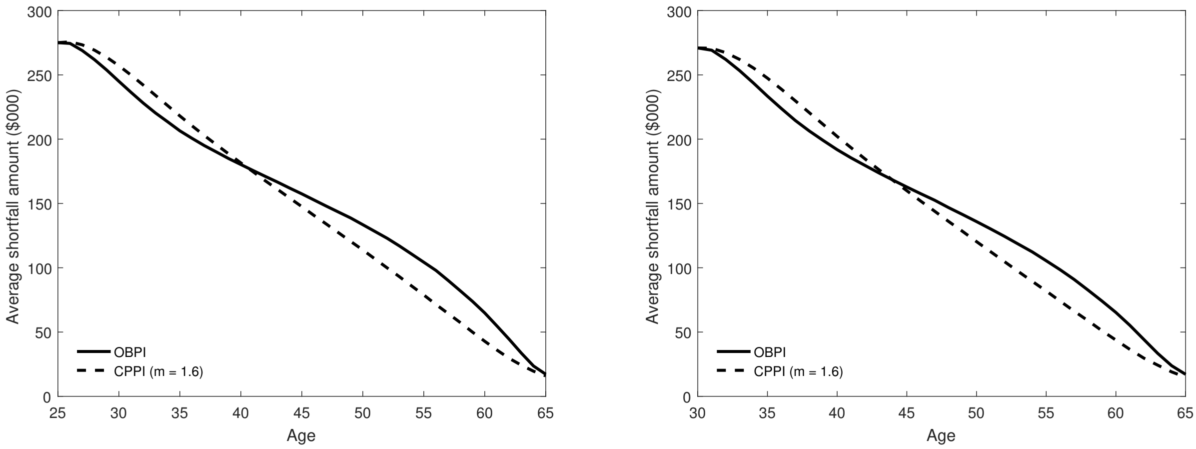


Figure 4. The absolute value of average shortfall amount: (Left Panel) members join the fund at age 25; (Right Panel) members join the fund at age 30. Average shortfall amount is given by $\mathbb{E}\left[X_T - A_T | X_T < A_T\right]$.

Figure 6, respectively. The portfolio weights in the bond fund move almost in the opposite direction to those in the equity fund, showing an upward trend for the younger cohorts and a downward trend for the older cohorts. The average portfolio weights in the cash fund show less variation across the different cohorts.

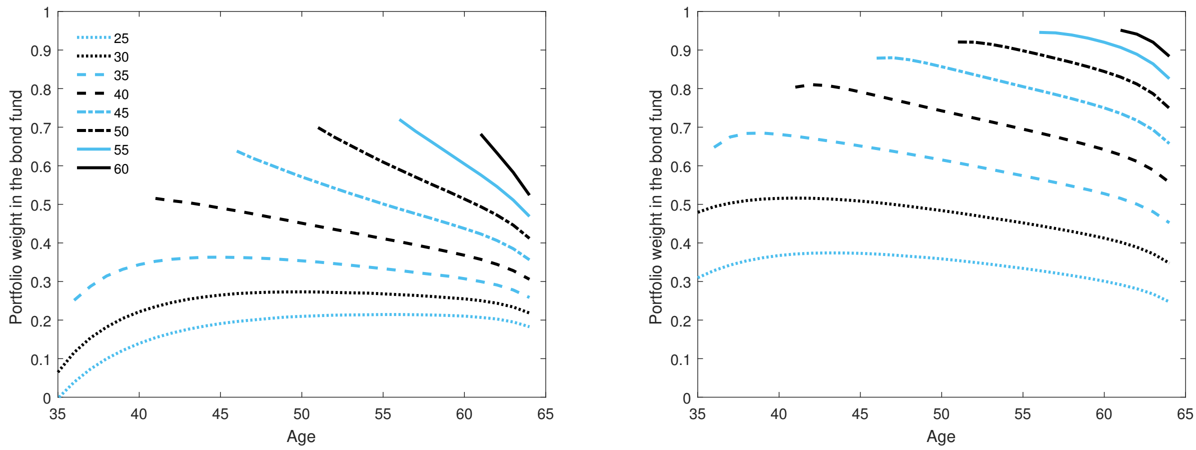


Figure 5. Average portfolio weights in the bond fund: (Left Panel) the OBPI strategy, (Right Panel) the CPPI strategy with $m = 1.6$.

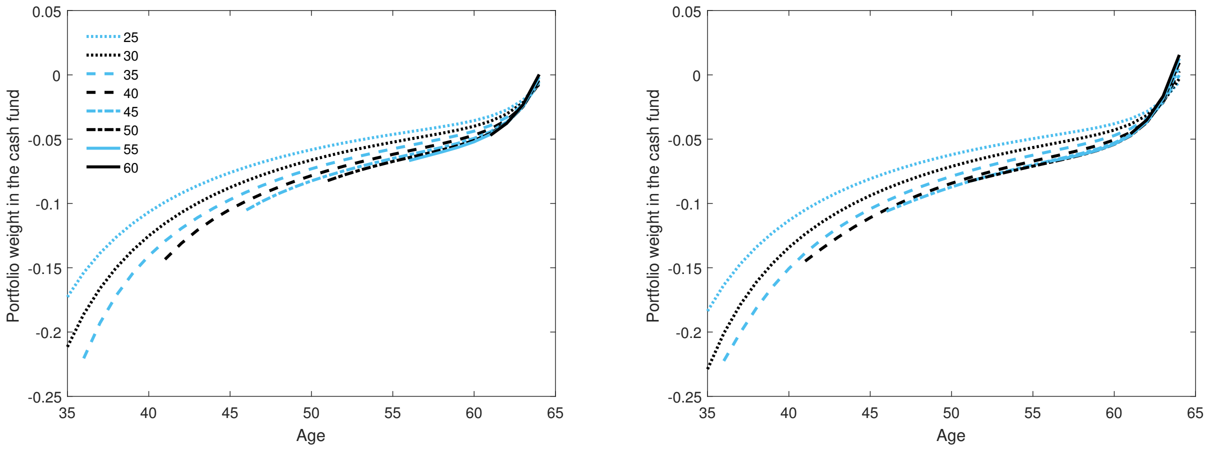


Figure 6. Average portfolio weights in the cash fund: (Left Panel) the OBPI strategy, (Right Panel) the CPPI strategy with $m = 1.6$.

To understand the differences between the portfolio insurance strategies better we investigate the sample paths of the asset weights for fund members who are 25 at time 0. We choose the youngest cohort since they have the longest investment horizon. Once again, we focus on the results from age 35 onwards, for the portfolio weights in early years are volatile due to the low fund balance. Figure 7 shows sample paths and the 95% confidence intervals for the portfolio weights in the equity fund. Although the average weights decrease over age, they have wide confidence bounds reflecting equity fund volatility. For the CPPI strategy, the 95% confidence intervals become significantly wider as the multiplier increases. Figure 8 shows these for multipliers of 1.2 and 2.0.

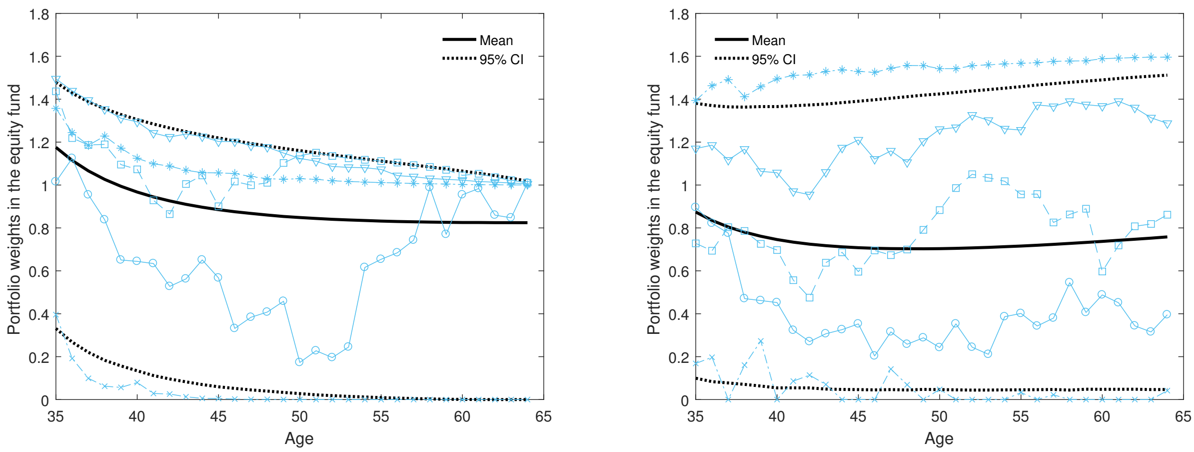


Figure 7. Some simulated sample paths (blue lines with markers), mean and 95% confidence intervals of portfolio weights in the equity fund: (Left Panel) the OBPI strategy; (Right Panel) the CPPI strategy with $m = 1.6$. The member joins the fund at age 25.

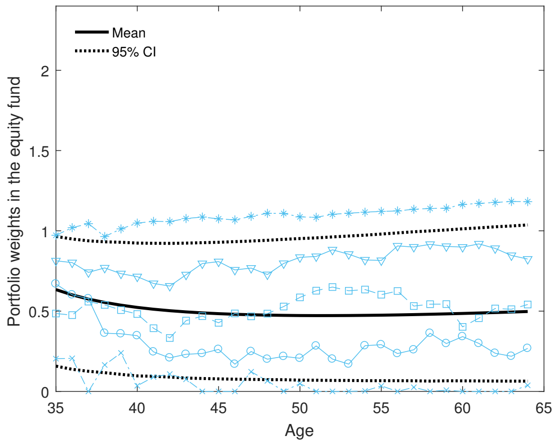


Figure 8. Some simulated sample paths (blue lines with markers), mean and 95% confidence intervals of portfolio weights in the equity fund: (Left Panel) the CPPI strategy with $m = 1.2$; (Right Panel) the CPPI strategy with $m = 2$. The member joins the fund at age 25.

5.2 Comparison of retirement payoffs

To further evaluate and compare the effectiveness of the portfolio insurance strategies, we examine the portfolio values at retirement. Table 6 summarises the mean, median, 95% confidence intervals, shortfall probability, and average shortfall amount of the portfolio values at retirement. The shortfall reflects the annual rebalancing assumption, as no shortfall would occur under the assumption of continuous rebalancing. In practice, it would be possible to rebalance the portfolio more frequently (e.g. monthly or weekly), but this would incur higher transaction costs. We do not explicitly consider transaction costs in the study, but it is worth noting that the transaction costs could be very substantial (Boyle and Vorst, 1992). As a consequence, the shortfall probability would not necessarily decrease if the rebalancing frequency was increased.

It is noticeable that the average shortfall amount remains less than 3% of the target annuitization level across different cohorts, demonstrating the effectiveness of the strategy in meeting the target. Comparing the two strategies within each single cohort, the CPPI strategy provides a better downside risk protection as indicated by significantly lower shortfall probabilities. The average shortfall amount is comparable between the two strategies, though. In terms of the fund balance at retirement, the OBPI strategy usually gives a higher average amount.

Increasing the value of the CPPI multiplier improves the average payoff at the cost of weaker downside protection. The median payoff is hardly affected by the multiplier. In terms of the inter-cohort differences, the older the cohort, the lower the payoff level, and consequently the

Table 6. Mean, median, 95% confidence intervals (CI), shortfall probability, and average shortfall amount of the portfolio values at retirement.

	Mean ($000)	Median ($000)	95% CI ($000)	Shortfall probability	Average shortfall ($000)	(A_T)
25 years old at time 0						
X_T^{OBPI}	2,616	1,563	(598, 11,223)	0.081	−17.19	−0.027
$X_T^{\text{CPPI}}(m = 1.2)$	1,427	1,023	(660, 4,720)	0.004	−13.85	−0.021
$X_T^{\text{CPPI}}(m = 1.6)$	2,396	1,163	(643, 11,892)	0.010	−15.93	−0.025
$X_T^{\text{CPPI}}(m = 2.0)$	4,445	1,240	(627, 27,280)	0.022	−17.68	−0.027
30 years old at time 0						
X_T^{OBPI}	2,076	1,305	(596, 8,184)	0.098	−17.11	−0.027
$X_T^{\text{CPPI}}(m = 1.2)$	1,157	912	(651, 3,165)	0.006	−13.27	−0.020
$X_T^{\text{CPPI}}(m = 1.6)$	1,697	1,001	(638, 7,057)	0.013	−14.89	−0.023
$X_T^{\text{CPPI}}(m = 2.0)$	2,736	1,054	(624, 14,991)	0.026	−17.17	−0.027
35 years old at time 0						
X_T^{OBPI}	1,664	1,106	(593, 6,000)	0.115	−17.16	−0.027
$X_T^{\text{CPPI}}(m = 1.2)$	975	829	(645, 2,186)	0.009	−12.41	−0.019
$X_T^{\text{CPPI}}(m = 1.6)$	1,270	885	(633, 4,261)	0.018	−14.63	−0.023
$X_T^{\text{CPPI}}(m = 2.0)$	1,784	919	(622, 8,202)	0.032	−16.58	−0.026
40 years old at time 0						
X_T^{OBPI}	1,353	940	(592, 4,401)	0.135	−16.86	−0.027
$X_T^{\text{CPPI}}(m = 1.2)$	853	769	(639, 1,583)	0.014	−13.25	−0.020
$X_T^{\text{CPPI}}(m = 1.6)$	1,010	804	(629, 2,664)	0.024	−15.05	−0.023
$X_T^{\text{CPPI}}(m = 2.0)$	1,259	826	(620, 4,613)	0.039	−16.74	−0.026
45 years old at time 0						
X_T^{OBPI}	1,123	816	(590, 3,258)	0.156	−16.86	−0.027
$X_T^{\text{CPPI}}(m = 1.2)$	773	724	(634, 1,199)	0.022	−12.80	−0.020
$X_T^{\text{CPPI}}(m = 1.6)$	853	745	(626, 1,746)	0.033	−14.67	−0.023
$X_T^{\text{CPPI}}(m = 2.0)$	971	759	(618, 2,650)	0.049	−16.28	−0.025
50 years old at time 0						
X_T^{OBPI}	948	724	(588, 2,374)	0.181	−17.05	−0.027
$X_T^{\text{CPPI}}(m = 1.2)$	718	692	(631, 957)	0.035	−12.65	−0.019
$X_T^{\text{CPPI}}(m = 1.6)$	757	704	(624, 1,208)	0.046	−14.17	−0.022
$X_T^{\text{CPPI}}(m = 2.0)$	808	712	(618, 1,594)	0.062	−15.70	−0.024
55 years old at time 0						
X_T^{OBPI}	821	675	(586, 1,710)	0.201	−17.21	−0.027
$X_T^{\text{CPPI}}(m = 1.2)$	682	669	(629, 808)	0.056	−11.92	−0.018
$X_T^{\text{CPPI}}(m = 1.6)$	698	675	(623, 913)	0.066	−13.02	−0.020
$X_T^{\text{CPPI}}(m = 2.0)$	719	679	(618, 1,058)	0.080	−14.25	−0.022
60 years old at time 0						
X_T^{OBPI}	725	656	(585, 1,188)	0.217	−17.40	−0.028
$X_T^{\text{CPPI}}(m = 1.2)$	657	652	(629, 714)	0.091	−11.84	−0.018
$X_T^{\text{CPPI}}(m = 1.6)$	663	655	(624, 749)	0.095	−12.47	−0.019
$X_T^{\text{CPPI}}(m = 2.0)$	669	657	(619, 791)	0.104	−13.13	−0.020

higher the chance of falling short of the target. This is due to the lower present values of future contributions for older cohorts even though they contribute higher initial fund balances.

Figure 9 shows a payoff diagram to compare the portfolio insurance strategies. To facilitate the measurement against the target, both the payoff (y-axis) and the equity fund price (x-axis) are denominated in the target annuitization level, A_T. A payoff below one means that the target is not met. The figure reinforces the results in Table 6. The OBPI strategy generally performs better unless the equity fund performs poorly. In addition, the OBPI strategy has more outcomes below the target annuitization level, consistent with the result that the OBPI strategy has a much higher shortfall probability than the CPPI (Table 6). These results hold for a lower value of the multiplier (Figure 10 (a)), and a shorter investment horizon (Figure 10 (c)). If the multiplier increases (Figure 10 (b)), however, the payoff of the CPPI strategy is no longer dominated by that of the OBPI strategy, and it also has a wider distribution.

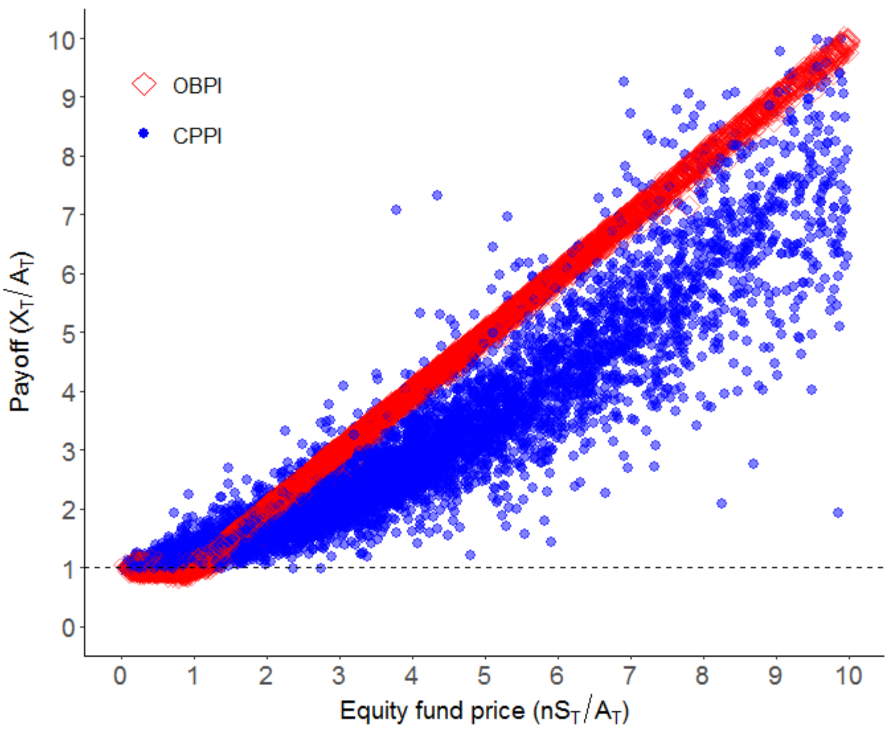
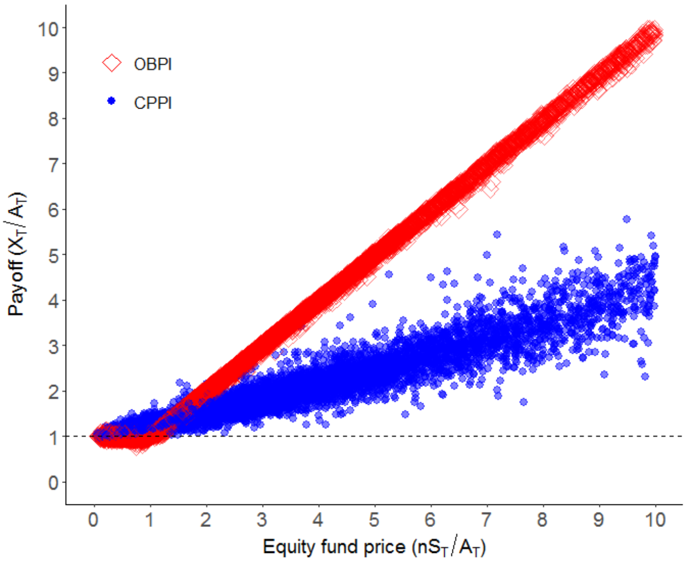


Figure 9. Comparison of OBPI and CPPI payoffs. The member joins the fund at age 25 and the CPPI multiplier (m) is 1.6. The dashed horizontal line through 1 on the y-axis represents the target annuitization level.

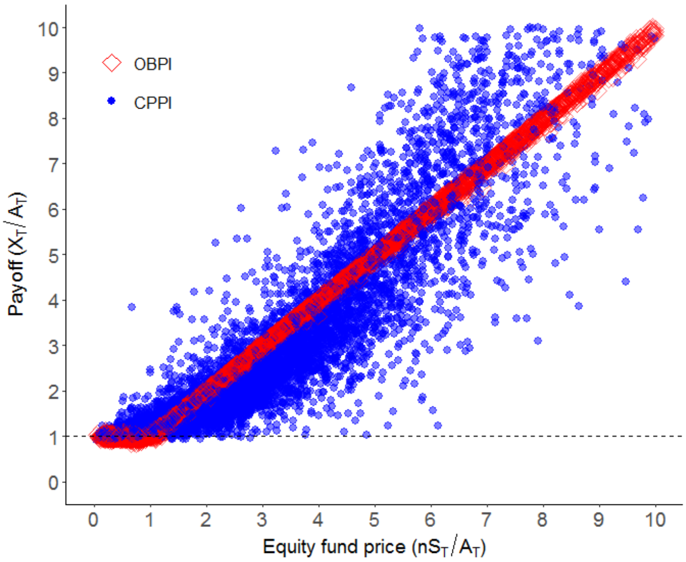
5.3 Alternative assumptions

We have shown that CPPI performs better than OBPI in downside risk protection, and that the average shortfall amount is less than 3% of the target annuitization level at retirement for both

(a) 25 years old at time 0, the CPPI multiplier is 1.2



(b) 25 years old at time 0, the CPPI multiplier is 2.0



(c) 50 years old at time 0, the CPPI multiplier is 1.6

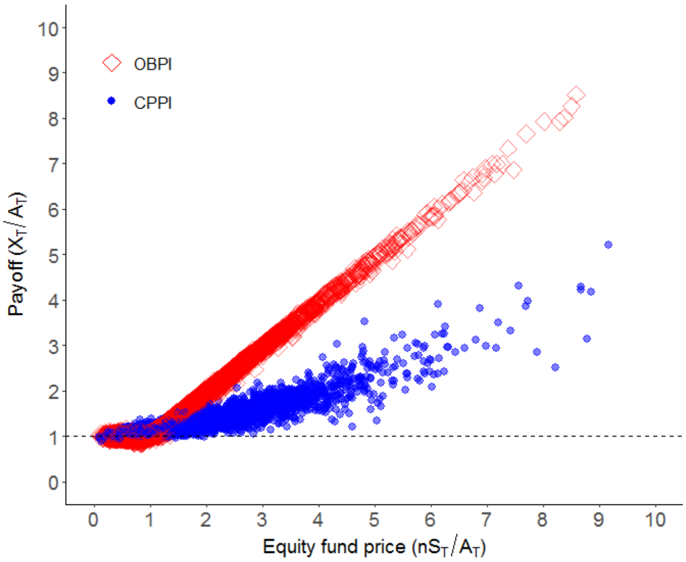


Figure 10. Comparison of OBPI and CPPI payoffs in selected scenarios: (a) the CPPI multiplier is close to 1; (b) the CPPI multiplier is relatively large; (c) the investment horizon is relatively short. The dashed horizontal line through 1 on the y-axis represents the target annuitization level.

strategies across different cohorts. We will consider alternative assumptions and their impact on both strategies. In particular, we investigate the extent to which the shortfall probability and the average shortfall amount changes if *1*) the equity fund price is more volatile, or *2*) the initial fund balance, X_0, is lower. We focus on the equity volatility since the portfolio weights on the equity fund are the main focus of our analysis.

5.3.1 Equity fund volatility

The U.S. stock market experienced noticeably high volatility during the Great Depression (1929–1939), October 1987, and the financial crisis in 2008. While the high levels of volatility in 1987 and 2008 were short-lived, the one during the Great Depression was prolonged (Schwert, 2011). Since the pension fund investment is long-term horizon, we consider the impact of a stock volatility of the Great Depression and set σ_S to 30%, which is similar to the standard deviation of the U.S. stock market over that period (Schwert, 2011).

Figure 11 shows that the CPPI strategy is far more sensitive to the change in the equity fund volatility. Compared to Figure 9, the payoffs of the CPPI strategy are more widespread, resulting in more outcomes below the target level. By contrast, there are no significant changes to the payoffs of the OBPI strategy.

Table 7 also shows that the shortfall probability and average shortfall amount, measuring the downside protection provided, are less robust for the CPPI strategy compared to the OBPI. For members joining the fund at age 25, the shortfall probability of the CPPI strategy grows more than tenfold while that of the OBPI strategy increases by approximately 80%. These increases are small at the older entry ages but still significant. Overall the average shortfall amount remains a small proportion of the target annuitization level at retirement. Except for the cases where the CPPI multiplier is large and the exposure to the high volatility spans a few decades, the average shortfall amount conditional on its occurrence is less than 5% of the target.

Table 7. The shortfall probability and average shortfall amount by different volatility levels of the equity fund.

	Shortfall probability		Average shortfall ($000)		(A_T)	
σ_S	0.157	0.30	0.157	0.30	0.157	0.30
25 years old at time 0						
X_T^{OBPI}	0.081	0.145	−17.19	−19.21	−0.027	−0.031
$X_T^{\text{CPPI}}(m = 1.2)$	0.004	0.042	−13.85	−19.54	−0.021	−0.030
$X_T^{\text{CPPI}}(m = 1.6)$	0.010	0.111	−15.93	−41.59	−0.025	−0.066
$X_T^{\text{CPPI}}(m = 2.0)$	0.022	0.284	−17.68	−957.75	−0.027	−1.543
30 years old at time 0						
X_T^{OBPI}	0.098	0.166	−17.11	−19.01	−0.027	−0.030
$X_T^{\text{CPPI}}(m = 1.2)$	0.006	0.048	−13.27	−19.14	−0.020	−0.030
$X_T^{\text{CPPI}}(m = 1.6)$	0.013	0.116	−14.89	−29.78	−0.023	−0.047
$X_T^{\text{CPPI}}(m = 2.0)$	0.026	0.271	−17.17	−313.39	−0.027	−0.510
35 years old at time 0						
X_T^{OBPI}	0.115	0.184	−17.16	−18.95	−0.027	−0.030
$X_T^{\text{CPPI}}(m = 1.2)$	0.009	0.056	−12.41	−18.35	−0.019	−0.029
$X_T^{\text{CPPI}}(m = 1.6)$	0.018	0.122	−14.63	−24.28	−0.023	−0.038
$X_T^{\text{CPPI}}(m = 2.0)$	0.032	0.259	−16.58	−168.81	−0.026	−0.273
40 years old at time 0						
X_T^{OBPI}	0.135	0.208	−16.86	−18.86	−0.027	−0.030
$X_T^{\text{CPPI}}(m = 1.2)$	0.014	0.066	−13.25	−17.92	−0.020	−0.028
$X_T^{\text{CPPI}}(m = 1.6)$	0.024	0.128	−15.05	−22.41	−0.023	−0.035
$X_T^{\text{CPPI}}(m = 2.0)$	0.039	0.248	−16.74	−80.10	−0.026	−0.129
45 years old at time 0						
X_T^{OBPI}	0.156	0.229	−16.86	−18.90	−0.027	−0.030
$X_T^{\text{CPPI}}(m = 1.2)$	0.022	0.076	−12.80	−17.47	−0.020	−0.027
$X_T^{\text{CPPI}}(m = 1.6)$	0.033	0.135	−14.67	−20.73	−0.023	−0.032
$X_T^{\text{CPPI}}(m = 2.0)$	0.049	0.238	−16.28	−47.90	−0.025	−0.076
50 years old at time 0						
X_T^{OBPI}	0.181	0.254	−17.05	−19.09	−0.027	−0.030
$X_T^{\text{CPPI}}(m = 1.2)$	0.035	0.090	−12.65	−16.77	−0.019	−0.026
$X_T^{\text{CPPI}}(m = 1.6)$	0.046	0.143	−14.17	−19.51	−0.022	−0.030
$X_T^{\text{CPPI}}(m = 2.0)$	0.062	0.229	−15.70	−29.60	−0.024	−0.047
55 years old at time 0						
X_T^{OBPI}	0.201	0.273	−17.21	−19.08	−0.027	−0.030
$X_T^{\text{CPPI}}(m = 1.2)$	0.056	0.107	−11.92	−15.26	−0.018	−0.024
$X_T^{\text{CPPI}}(m = 1.6)$	0.066	0.151	−13.02	−17.78	−0.020	−0.028
$X_T^{\text{CPPI}}(m = 2.0)$	0.080	0.215	−14.25	−21.91	−0.022	−0.034
60 years old at time 0						
X_T^{OBPI}	0.217	0.285	−17.40	−20.04	−0.028	−0.032
$X_T^{\text{CPPI}}(m = 1.2)$	0.091	0.125	−11.84	−13.89	−0.018	−0.021
$X_T^{\text{CPPI}}(m = 1.6)$	0.095	0.156	−12.47	−15.49	−0.019	−0.024
$X_T^{\text{CPPI}}(m = 2.0)$	0.104	0.195	−13.13	−17.54	−0.020	−0.027

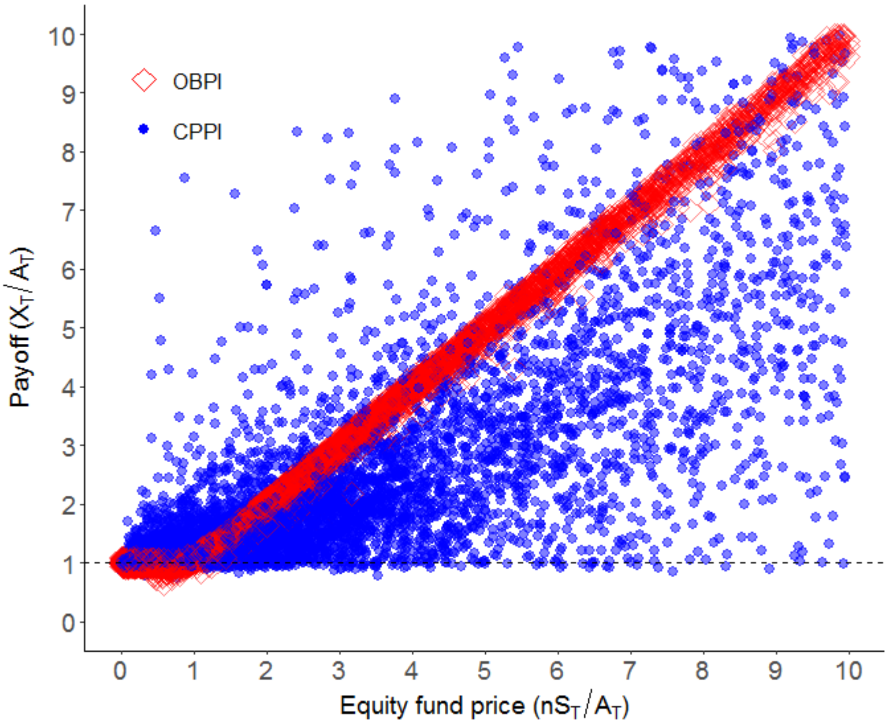


Figure 11. Comparison of OBPI and CPPI payoffs when $\sigma_S = 0.3$. The member joins the fund at age 25 and the CPPI multiplier (m) is 1.6. The dashed horizontal line through 1 on the y-axis represents the target annuitization level.

5.3.2 Initial fund contributions

The initial fund balance was determined so that the value of the self-financing portfolio, Y_0, was \$33,000 above the target. We now decrease the buffer above the target to \$10,000. The resulting initial fund balance (X_0) for each cohort is shown in Table 8. Note that for members joining the fund at age 25, the present value of future contributions is about \$32,760 above the target, A_0. Setting the buffer above the target at \$10,000 makes the initial fund balance negative. We therefore set the initial fund balance for the youngest cohort to zero.

Table 8. Initial fund balance (\$000) for each cohort.

25	30	35	40	45	50	55	60
0.00	9.95	56.26	116.05	189.25	275.84	375.86	489.41

Figure 12 compares the payoffs for members joining the fund at age 50. We choose an older cohort because their balances at retirement are more severely affected by lower initial balances. A lower initial fund balance limits the fund's ability to participate in the equity market, so both strategies take less advantage of better equity fund performance to improve the payoff. Figure 12 shows that the impact is stronger for the CPPI strategy. The OBPI strategy is more robust to a lower amount of initial contribution to the fund.

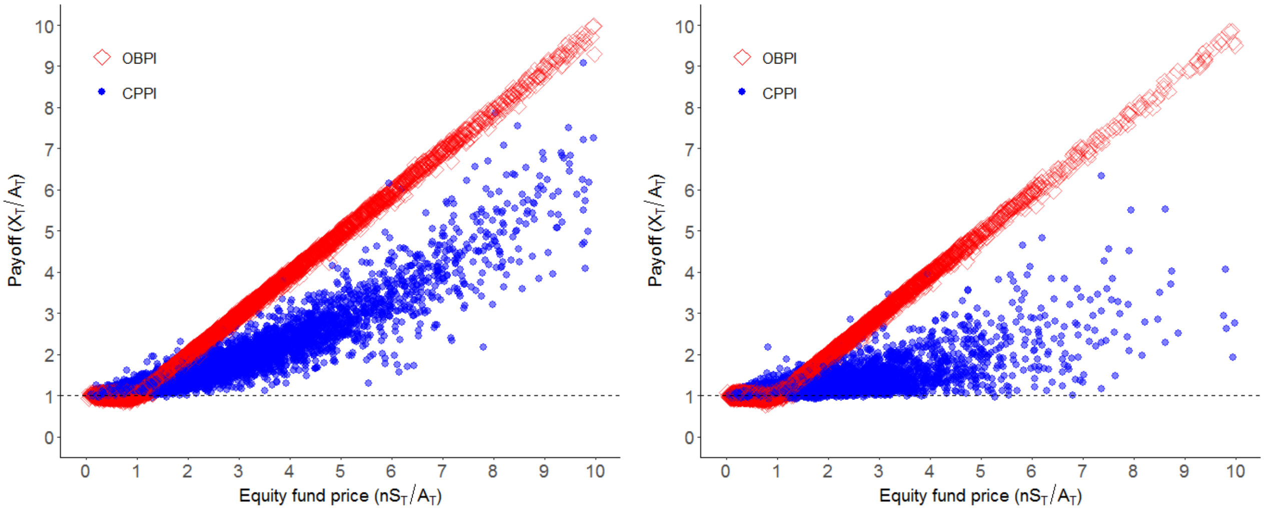


Figure 12. Comparison of payoffs between different levels of initial fund balances: (Left Panel) base case; (Right Panel) lower initial fund balance. The member joins the fund at age 40 and the CPPI multiplier (m) is 1.6. The dashed horizontal line through 1 on the y-axis represents the target annuitization level.

Table 9 compares the shortfall probability and the average shortfall amount. The impact on the youngest cohort is minimal since their initial fund balance is reduced by \$240 only compared to the base case. For the other cohorts, the shortfall probability increases, especially for the CPPI strategy. For the original assumptions, the OBPI strategy leads to a higher shortfall probability across all cohorts. When the initial fund balance is reduced, the OBPI strategy gives lower shortfall probabilities for older cohorts.

Another difference compared to the original case is that the shortfall probability of the CPPI strategy decreases rather than increases with the multiplier if members join the fund after mid-30s. This can be explained by examining the shortfall probability over the course of accumulation period. Figure 13 shows that in the first few years after joining the fund, the CPPI strategy with the lowest multiplier has the lowest rate of reduction in the shortfall probability. As members approach retirement, however, this ranking is reversed because the shortfall probability with a lower multiplier declines at a faster rate. The higher the initial fund balance, the earlier the change occurs. Given enough time, the CPPI strategy with a lower multiplier will result in a lower shortfall probability.

In terms of the changes in the absolute value of the average shortfall, the OBPI strategy shows a slight decrease, whereas the CPPI strategy shows some increases. Reducing the initial fund

Table 9. The shortfall probability and average shortfall amount by different initial fund balances (X_0). The initial fund balance for the original assumptions is shown in Table 4, and the one for the 'Lower' case is shown in Table 8.

X_0	Shortfall probability Original	Shortfall probability Lower	Average shortfall (\$000) Original	Average shortfall (\$000) Lower	Average shortfall (A_T) Original	Average shortfall (A_T) Lower
25 years old at time 0						
X_T^{OBPI}	0.081	0.082	−17.19	−17.19	−0.027	−0.027
$X_T^{\text{CPPI}}(m = 1.2)$	0.004	0.005	−13.85	−13.83	−0.021	−0.021
$X_T^{\text{CPPI}}(m = 1.6)$	0.010	0.011	−15.93	−15.91	−0.025	−0.025
$X_T^{\text{CPPI}}(m = 2.0)$	0.022	0.022	−17.68	−17.71	−0.027	−0.027
30 years old at time 0						
X_T^{OBPI}	0.098	0.184	−17.11	−16.58	−0.027	−0.026
$X_T^{\text{CPPI}}(m = 1.2)$	0.006	0.075	−13.27	−15.52	−0.020	−0.024
$X_T^{\text{CPPI}}(m = 1.6)$	0.013	0.080	−14.89	−17.24	−0.023	−0.027
$X_T^{\text{CPPI}}(m = 2.0)$	0.026	0.093	−17.17	−19.05	−0.027	−0.030
35 years old at time 0						
X_T^{OBPI}	0.115	0.206	−17.16	−16.56	−0.027	−0.026
$X_T^{\text{CPPI}}(m = 1.2)$	0.009	0.107	−12.41	−15.80	−0.019	−0.024
$X_T^{\text{CPPI}}(m = 1.6)$	0.018	0.107	−14.63	−17.18	−0.023	−0.027
$X_T^{\text{CPPI}}(m = 2.0)$	0.032	0.117	−16.58	−18.76	−0.026	−0.029
40 years old at time 0						
X_T^{OBPI}	0.135	0.234	−16.86	−16.54	−0.027	−0.026
$X_T^{\text{CPPI}}(m = 1.2)$	0.014	0.152	−13.25	−15.98	−0.020	−0.025
$X_T^{\text{CPPI}}(m = 1.6)$	0.024	0.145	−15.05	−17.08	−0.023	−0.026
$X_T^{\text{CPPI}}(m = 2.0)$	0.039	0.149	−16.74	−18.49	−0.026	−0.029
45 years old at time 0						
X_T^{OBPI}	0.156	0.258	−16.86	−16.62	−0.027	−0.026
$X_T^{\text{CPPI}}(m = 1.2)$	0.022	0.209	−12.80	−16.14	−0.020	−0.025
$X_T^{\text{CPPI}}(m = 1.6)$	0.033	0.193	−14.67	−17.08	−0.023	−0.026
$X_T^{\text{CPPI}}(m = 2.0)$	0.049	0.192	−16.28	−18.08	−0.025	−0.028
50 years old at time 0						
X_T^{OBPI}	0.181	0.288	−17.05	−16.51	−0.027	−0.026
$X_T^{\text{CPPI}}(m = 1.2)$	0.035	0.282	−12.65	−16.48	−0.019	−0.025
$X_T^{\text{CPPI}}(m = 1.6)$	0.046	0.258	−14.17	−17.12	−0.022	−0.027
$X_T^{\text{CPPI}}(m = 2.0)$	0.062	0.249	−15.70	−17.92	−0.024	−0.028
55 years old at time 0						
X_T^{OBPI}	0.201	0.312	−17.21	−16.46	−0.027	−0.026
$X_T^{\text{CPPI}}(m = 1.2)$	0.056	0.363	−11.92	−17.02	−0.018	−0.026
$X_T^{\text{CPPI}}(m = 1.6)$	0.066	0.338	−13.02	−17.34	−0.020	−0.027
$X_T^{\text{CPPI}}(m = 2.0)$	0.080	0.324	−14.25	−17.73	−0.022	−0.028
60 years old at time 0						
X_T^{OBPI}	0.217	0.336	−17.40	−16.54	−0.028	−0.026
$X_T^{\text{CPPI}}(m = 1.2)$	0.091	0.441	−11.84	−17.55	−0.018	−0.027
$X_T^{\text{CPPI}}(m = 1.6)$	0.095	0.424	−12.47	−17.65	−0.019	−0.027
$X_T^{\text{CPPI}}(m = 2.0)$	0.104	0.410	−13.13	−17.82	−0.020	−0.028

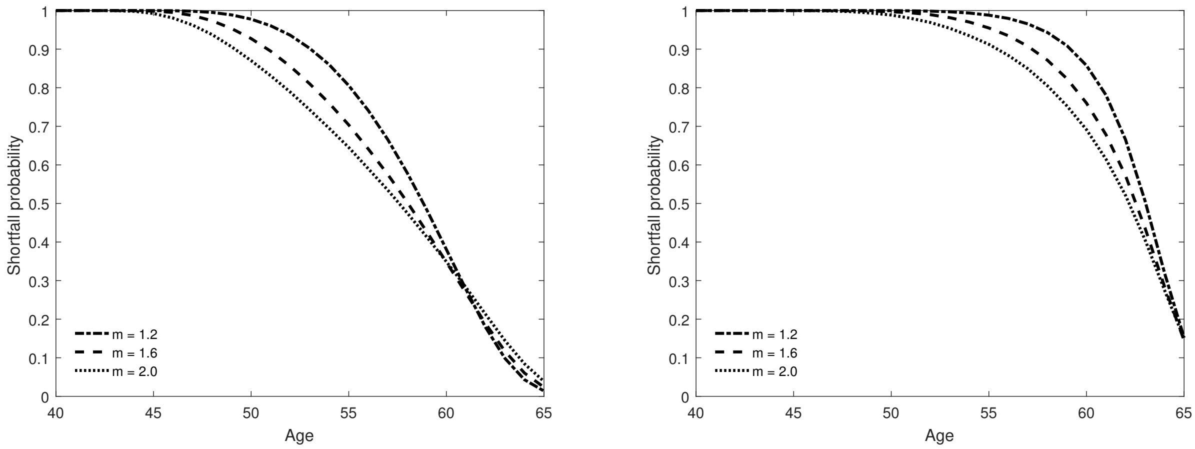


Figure 13. Comparison of shortfall probability between different levels of initial fund balances: (Left Panel) original; (Right Panel) lower initial fund balance. The member joins the fund at age 40. Probability of shortfall is given by $\mathbb{P}(X_T < A_T)$.

balance decreases the standard deviations of the fund balances at retirement for both strategies, as shown in (Table 10), because the fund manager invests a lower proportion of wealth in the equity fund. However, the reduction in the initial fund balance has a different impact on the downside deviation between the two strategies. The downside deviation is given by

$$\sqrt{\mathbb{E}\Big[(X_T - A_T)^2 | X_T < A_T\Big]}. \quad (27)$$

Table 10 shows that the downside deviation of the OBPI strategy decreases, whereas that of the CPPI strategy increases by a relatively large percentage. As a result, when a shortfall occurs, the CPPI strategy leads to a worse average shortfall amount. This effect is larger at the older entry ages into the fund despite these ages having higher annual contributions.

6 Conclusions

The design of current DC pension funds usually has insufficient integration between the accumulation and retirement phases. This has been an issue since the accumulated wealth may not be able to provide retirees with a sustainable income level. Target annuitization funds aim to provide fund members with an amount of retirement benefits that can finance a desired post-retirement consumption within a confidence interval. They are a possible solution to connecting the accumulation and retirement phases in the DC pension plans, and have attracted increasing attention. Portfolio insurance strategies are suitable investment strategies to manage the target

Table 10. The standard deviation and the downside deviation (given in Equation (27)) by different initial fund balances (X_0). The initial fund balance for the 'Original' is shown in Table 4, and the one for the 'Lower' case is shown in Table 8.

	Standard deviation			Downside deviation		
X_0	Original ($000)	Lower ($000)	Difference (%)	Original ($000)	Lower ($000)	Difference (%)
25 years old at time 0						
X_T^{OBPI}	3,340	3,328	−0.4	22	22	0.0
$X_T^{\text{CPPI}}(m = 1.2)$	1,401	1,390	−0.8	18	18	0.0
$X_T^{\text{CPPI}}(m = 1.6)$	5,345	5,305	−0.7	21	21	0.0
$X_T^{\text{CPPI}}(m = 2.0)$	19,997	19,848	−0.7	23	23	0.1
30 years old at time 0						
X_T^{OBPI}	2,355	1,407	−40.2	23	21	−5.3
$X_T^{\text{CPPI}}(m = 1.2)$	832	271	−67.5	18	20	13.4
$X_T^{\text{CPPI}}(m = 1.6)$	2,785	923	−66.9	20	22	12.3
$X_T^{\text{CPPI}}(m = 2.0)$	9,294	3,188	−65.7	22	24	8.9
35 years old at time 0						
X_T^{OBPI}	1,648	1,001	−39.2	22	21	−4.8
$X_T^{\text{CPPI}}(m = 1.2)$	487	163	−66.5	17	20	21.1
$X_T^{\text{CPPI}}(m = 1.6)$	1,404	476	−66.1	19	22	14.2
$X_T^{\text{CPPI}}(m = 2.0)$	3,922	1,369	−65.1	22	24	10.7
40 years old at time 0						
X_T^{OBPI}	1,142	703	−38.5	22	21	−2.5
$X_T^{\text{CPPI}}(m = 1.2)$	282	97	−65.5	18	21	17.7
$X_T^{\text{CPPI}}(m = 1.6)$	712	243	−65.8	20	22	11.1
$X_T^{\text{CPPI}}(m = 2.0)$	1,740	593	−65.9	22	24	7.9
45 years old at time 0						
X_T^{OBPI}	784	486	−38.1	22	21	−2.7
$X_T^{\text{CPPI}}(m = 1.2)$	163	59	−63.9	17	21	19.9
$X_T^{\text{CPPI}}(m = 1.6)$	361	128	−64.6	20	22	12.5
$X_T^{\text{CPPI}}(m = 2.0)$	767	270	−64.9	21	23	8.0
50 years old at time 0						
X_T^{OBPI}	514	317	−38.3	22	22	−4.2
$X_T^{\text{CPPI}}(m = 1.2)$	90	35	−61.5	17	21	25.0
$X_T^{\text{CPPI}}(m = 1.6)$	173	64	−62.7	19	22	17.0
$X_T^{\text{CPPI}}(m = 2.0)$	316	115	−63.6	21	23	11.3
55 years old at time 0						
X_T^{OBPI}	318	193	−39.1	23	21	−5.9
$X_T^{\text{CPPI}}(m = 1.2)$	48	20	−58.3	16	22	36.4
$X_T^{\text{CPPI}}(m = 1.6)$	81	33	−59.7	17	22	27.7
$X_T^{\text{CPPI}}(m = 2.0)$	129	51	−60.7	19	22	20.3
60 years old at time 0						
X_T^{OBPI}	164	96	−41.6	23	22	−7.3
$X_T^{\text{CPPI}}(m = 1.2)$	22	10	−54.8	16	22	41.6
$X_T^{\text{CPPI}}(m = 1.6)$	33	14	−56.3	16	22	35.5
$X_T^{\text{CPPI}}(m = 2.0)$	45	19	−57.6	17	22	29.9

annuitization fund because they provide investors with the potential to limit downside risk and to participate on the upside.

We have analysed and compared the performance of OBPI and CPPI strategies for a target annuitization fund. We derive theoretical results for both and implement using simulations. Both strategies have similar patterns for the average portfolio weights in an equity fund. For members joining the fund before ages in the mid-30s, the portfolio weights in the equity fund tend to decrease as they get older, but the weights are volatile due to the equity market volatility. For members joining the fund at older ages, the average portfolio weights in the equity fund increase as they grow older. The difference reflects that the younger cohorts have larger values of future contributions, which are a form of safe asset.

In terms of downside risk protection, the average shortfall amount as a proportion of the target annuitization level at retirement is minimal for both strategies, and robust to a shorter accumulation period and lower contribution levels to the fund. A higher equity market volatility can significantly increase the shortfall amount if the CPPI multiplier is large and the exposure to the high volatility lasts for a long time period.

Our analysis shows the CPPI strategy performs significantly better in reducing the likelihood of shortfall, although this performance is more sensitive to the level of equity market volatility and the initial fund contributions. The OBPI strategy typically gives a higher portfolio value at retirement and its ability to provide downside risk protection is more robust to changing equity market volatility and the amount of initial fund contributions.

Both strategies have desirable outcomes for a DC pension fund aiming to provide a minimum level of retirement income to its members on reaching retirement age. Our analysis provides a basis for a fund to determine which strategy would suit its member profiles and also how to implement these strategies.

7 Acknowledgement

The authors acknowledge financial support from an ARC Discovery Grant DP170102275 Retirement Income Product Innovation, from CEPAR Australian Research Council Centre of Excellence in Population Ageing Research project number CE170100005 and the Society of Actuaries Center of Actuarial Excellence Research Grant 2017–2020: Longevity Risk: Actuarial and Predictive Models, Retirement Product Innovation, and Risk Management Strategies.

References

Association of Superannuation Funds of Australia (2017) *ASFA Retirement Standard detailed budget breakdowns March quarter 2017*. Report. ASFA Research and Resource Centre, Sydney.

Battocchio, P. and Menoncin, F. (2004) Optimal pension management in a stochastic framework. *Insurance: Mathematics and Economics*, **34**(1), 79–95.

Bernard, C. and Kwak, M. (2016) Dynamic preferences for popular investment strategies in pension funds. *Scandinavian Actuarial Journal*, **2016**(5), 398–419.

Binswanger, J. and Schunk, D. (2012) What is an adequate standard of living during retirement? *Journal of Pension Economics & Finance*, **11**(2), 203–222.

Black, F. and Jones, R. W. (1987) Simplifying portfolio insurance. *The Journal of Portfolio Management*, **14**(1), 48–51.

Black, F. and Perold, A. F. (1992) Theory of constant proportion portfolio insurance. *Journal of Economic Dynamics and Control*, **16**(3), 403–426.

Blake, D., Cairns, A. J. G. and Dowd, K. (2001) Pensionmetrics: Stochastic pension plan design and value-at-risk during the accumulation phase. *Insurance: Mathematics and Economics*, **29**(2), 187–215.

Blake, D., Cairns, A. J. G. and Dowd, K. (2008) *Turning pensions plans into pension planes: What investment strategy designers of defined contribution pension plans can learn from commercial aircraft designers*. Discussion Paper PI-0806. Pensions Institute.

Blake, D., Wright, D. and Zhang, Y. (2013) Target-driven investing: Optimal investment strategies in defined contribution pension plans under loss aversion. *Journal of Economic Dynamics and Control*, **37**(1), 195–209.

Blake, D., Wright, D. and Zhang, Y. (2014) Age-dependent investing: Optimal funding and investment strategies in defined contribution pension plans when members are rational life cycle financial planners. *Journal of Economic Dynamics and Control*, **38**, 105–124.

Boulier, J.-F., Huang, S. and Taillard, G. (2001) Optimal management under stochastic interest rates: The case of a protected defined contribution pension fund. *Insurance: Mathematics and Economics*, **28**(2), 173–189.

Boyle, P. P. and Vorst, T. (1992) Option replication in discrete time with transaction costs. *Journal of Finance*, **47**(1), 271–293.

Brennan, M. J. and Xia, Y. (2002) Dynamic asset allocation under inflation. *Journal of Finance*, **57**(3), 1201–1238.

Brown, J. R. (2009) Understanding the role of annuities in retirement planning. In: *Overcoming the Savings Slump: How to Increase the Effectiveness of Financial Education and Saving Programs*. Ed. by A. Lusardi. Chicago, IL: University of Chicago Press. Chap. 6, pp. 178–206.

Bureau of Labor Statistics, U.S. Department of Labor (2017) Labor productivity growth since the Great Recession. *The Economics Daily*. Available at https://www.bls.gov/opub/ted/2017/labor-productivity-growth-since-the-great-recession.htm (visited December 13, 2018).

Cairns, A. J. G., Blake, D. and Dowd, K. (2006) Stochastic lifestyling: Optimal dynamic asset allocation for defined contribution pension plans. *Journal of Economic Dynamics and Control*, **30**(5), 843–877.

Financial System Inquiry (2014) *Financial System Inquiry Final Report*. Canberra: Commonwealth of Australia.

Geman, H., El Karoui, N. and Rochet, J.-C. (1995) Changes of numeraire, changes of probability measure and option pricing. *Journal of Applied Probability*, **32**(2), 443–458.

Guan, G. and Liang, Z. (2014) Optimal management of DC pension plan in a stochastic interest rate and stochastic volatility framework. *Insurance: Mathematics and Economics*, **57**, 58–66.

Han, N.-w. and Hung, M.-w. (2012) Optimal asset allocation for DC pension plans under inflation. *Insurance: Mathematics and Economics*, **51**(1), 172–181.

Holzmann, R., Paul, R. and Dorfman, M. (2008) *Pension systems and reform conceptual framework*. Social Protection Discussion Papers No. 0824. The World Bank.

Impavido, G., Lasagabaster, E. and García-Huitrón, M. (2012) *New Policies for Defined Contribution Pensions: Industrial Organization Models and Investment Products*. Washington DC: The World Bank.

Jamshidian, F. (1989) An exact bond option formula. *Journal of Finance*, **44**(1), 205–209.

Leland, H. E. (1980) Who should buy portfolio insurance? *Journal of Finance*, **35**(2), 581–594.

Pézier, J. and Scheller, J. (2011) Optimal investment strategies and performance sharing rules for pension schemes with minimum guarantee. *Journal of Pension Economics and Finance*, **10**(1), 119–145.

Pézier, J. and Scheller, J. (2013) Best portfolio insurance for long-term investment strategies in realistic conditions. *Insurance: Mathematics and Economics*, **52**(2), 263–274.

Schwert, G. W. (2011) Stock volatility during the recent financial crisis. *European Financial Management*, **17**(5), 789–805.

Viceira, L. M. (2009) Life-cycle funds. In: *Overcoming the Saving Slump: How to Increase the Effectiveness of Financial Education and Saving Programs*. Ed. by A. Lusardi. Chicago, IL: University Of Chicago Press. Chap. 5, pp. 140–177.

Mengyi Xu (Corresponding author)
ARC Centre of Excellence in Population Ageing Research (CEPAR), UNSW Sydney
Level 3, East Wing, 223 Anzac Parade
Kensington, NSW 2033, Australia
E-mail: m.xu@unsw.edu.au

Appendix A Proofs in the option-based portfolio insurance strategy

A.1 Proof of Lemma 3.1

Jamshidian (1989) proves that pricing an option on a portfolio is equivalent to pricing a portfolio of options with appropriate strike prices, as long as the prices of the portfolio components are all strictly decreasing or increasing with the same state variable. In the present case, each component is a zero-coupon bond (denominated in the equity fund S), which is a monotonic function of $\varepsilon_{t,T}$. It is therefore possible to find an $\varepsilon^*_{t,T}$ such that

$$\sum_{j=0}^{J} P^{(S)}(t,T+j)\exp\left[\varepsilon^*_{t,T}\sqrt{\int_t^T \sigma_P^2(u,T+j)\mathrm{d}u} - \frac{1}{2}\int_t^T \sigma_P^2(u,T+j)\mathrm{d}u\right] = \frac{n}{g}.$$

Let $K_j^{(S)}(t)$ be the price of the zero-coupon bond (denominated in the equity fund S) that corresponds to $\varepsilon^*_{t,T}$, i.e.

$$K_j^{(S)}(t) = P^{(S)}(t,T+j)\exp\left[\varepsilon^*_{t,T}\sqrt{\int_t^T \sigma_P^2(u,T+j)\mathrm{d}u} - \frac{1}{2}\int_t^T \sigma_P^2(u,T+j)\mathrm{d}u\right].$$

Following the relationship between $\varepsilon_{t,T}$ and bond prices, it can be shown that

$$\left(\sum_{j=0}^{J} P^{(S)}(T,T+j) - \frac{n}{g}\right)^+ = \sum_{j=0}^{J}\left(P^{(S)}(T,T+j) - K_j^{(S)}(t)\right)^+,$$

which implies that

$$Q_t = \frac{g}{n}S_t\sum_{j=0}^{J}\widetilde{\mathbb{E}}_t^{(S)}\left[\left(P^{(S)}(T,T+j) - K_j^{(S)}(t)\right)^+\right].$$

A.2 Proof of Theorem 3.4

The differential of the portfolio is given by

$$\begin{aligned}\mathrm{d}\left(\frac{Q_t}{S_t}\right) = \frac{g}{n}\sum_{j=0}^{J}\Big[&N(-d_{2,t})\mathrm{d}P^{(S)}(t,T+j) + P^{(S)}(t,T+j)\mathrm{d}N(-d_{2,t}) \\ &+ \mathrm{d}P^{(S)}(t,T+j)\mathrm{d}N(-d_{2,t}) - K_j^{(S)}(t)\mathrm{d}N(-d_{1,t}) \\ &- N(-d_{1,t})\mathrm{d}K_j^{(S)}(t) - \mathrm{d}K_j^{(S)}(t)\mathrm{d}N(-d_{1,t})\Big].\end{aligned} \tag{28}$$

In order for the portfolio to be self-financing, the change of portfolio value needs to be entirely due to capital gains. In the following we will first show that

$$\sum_{j=0}^{J}\left[N(-d_{1,t})\mathrm{d}K_j^{(S)}(t) + \mathrm{d}K_j^{(S)}(t)\mathrm{d}N(-d_{1,t})\right] = 0, \tag{29}$$

and then show that for $j = 0, \cdots, J$

$$P^{(S)}(t, T+j)\mathrm{d}N(-d_{2,t}) + \mathrm{d}P^{(S)}(t, T+j)\mathrm{d}N(-d_{2,t}) - K_j^{(S)}(t)\mathrm{d}N(-d_{1,t}) = 0, \tag{30}$$

to prove the portfolio is self-financing.

To show Equation (29), we use the results of Remark 3.3 and Equation (17). In particular, the terms $N(-d_{1,t})$ and $\mathrm{d}N(-d_{1,t})$ do not depend on j due to Remark 3.3, and

$$\sum_{j=0}^{J} \mathrm{d}K_j^{(S)}(t) = \mathrm{d}\left(\sum_{j=0}^{J} K_j^{(S)}(t)\right) = \mathrm{d}\left(\frac{n}{g}\right) = 0,$$

due to Equation (17). Therefore, the left-hand side of Equation (29) is given by

$$N(-d_{1,t})\sum_{j=0}^{J} \mathrm{d}K_j^{(S)}(t) + \mathrm{d}N(-d_{1,t})\sum_{j=0}^{J} \mathrm{d}K_j^{(S)}(t) = 0.$$

To show Equation (30), we perform the following steps.

- Use the Itô's formula to derive $\mathrm{d}N(-d_{1,t})$, $\mathrm{d}N(-d_{2,t})$, and $\mathrm{d}P^{(S)}(t, T+j)\mathrm{d}N(-d_{2,t})$.
- Substitute the derivatives back to the left-hand side of Equation (30). The left-hand side consists of functions of $\mathrm{d}t$ and $\mathrm{d}(d_{1,t})$ only.
- Calculate the coefficients of $\mathrm{d}t$ and $\mathrm{d}(d_{1,t})$. Both turn out to be zero.

Hence the left-hand side of Equation (30) is equal to the right-hand side. Substitute Equations (29) and (30) back to Equation (28). The differential of the portfolio becomes

$$\mathrm{d}\left(\frac{Q_t}{S_t}\right) = \frac{g}{n}\sum_{j=0}^{J}\left[N(-d_{2,t})\mathrm{d}P^{(S)}(t, T+j)\right].$$

On the other hand, the capital gains differential associated with this portfolio, denominated in units of equity fund, is

$$\frac{g}{n}\sum_{j=0}^{J}\left[N(-d_{2,t})\mathrm{d}P^{(S)}(t, T+j)\right].$$

Therefore, the change of value in the portfolio is entirely due to capital gains. This proves Theorem 3.4 that the portfolio is self-financing.