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#### **Individual Post-Retirement Longevity Risk Management Under Systematic Mortality Risk**

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# Individual post-retirement longevity risk management under systematic mortality risk

Katja Hanewald\*, John Piggott<sup>†</sup> and Michael Sherris<sup>‡</sup>

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## Abstract

This paper analyzes an individual's post retirement longevity risk management strategy allowing for systematic longevity risk, recent product innovations, and product loadings. A complete-markets discrete state model and multi-period simulations of portfolio strategies are used to assess individual longevity insurance product portfolios with differing levels of systematic and idiosyncratic longevity risk. Portfolios include: fixed life annuities, deferred annuities, inflation-indexed annuities, phased withdrawals and recently proposed group self-annuitization (GSA) plans. GSA plans are found to replace even inflation-indexed annuity products when there are loadings on guaranteed life annuity products. With a bequest motive and loadings, coinsurance portfolio strategies with phased withdrawals and GSA's dominate portfolios with life annuities or deferred annuities.

**Keywords:** longevity risk, optimal insurance, life annuity, group self-annuitization (GSA), market frictions

**JEL Classifications:** D14, E21, G22, G23

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# 1 Introduction

Most developed countries use subsidies and tax incentives to increase mandatory and voluntary retirement savings so individuals are less reliant on public pay-as-you-go pension systems. Payout phases of pension systems are organized very differently across countries with individuals increasingly responsible for post-retirement financial decisions (Rocha *et al.*, 2010; Rocha and Vittas, 2010). Although most private retirement savings are in defined contribution plans, limited attention has been paid to the decumulation of these funds. Individuals face a complex problem of post-retirement financial planning. They have to take into account investment risk, inflation risk, product loadings and guarantees and both systematic and unsystematic (idiosyncratic) longevity risk.<sup>1</sup> Recent product innovations in the form of group self-annuitization (GSA) plans provide new opportunities to manage longevity risk.

The importance and complexity of the post-retirement financial planning problem has been well recognized in the literature. There is a longstanding literature on optimal annuitization dating back to (Yaari, 1965). Recent studies consider individuals' retirement portfolio choice with longevity insurance products such as life annuities and deferred annuities (e.g., Horneff *et al.*, 2010a; Post, 2010; Purcal and Piggott, 2008; Schulze and Post, 2010; Stevens, 2010), variable annuities (e.g., Doyle and Piggott, 2003; Milevsky and Kyrychenko, 2008; Horneff *et al.*, 2010b), or group self-annuitization plans (e.g., Piggott *et al.*, 2005; Valdez *et al.*, 2006; Stamos, 2008). Only a few of these papers distinguish between idiosyncratic and systematic longevity risk, model inflation risk or assess new product innovations including group self-annuitization.

This paper uses a simplified complete markets framework to study the optimal management of systematic and idiosyncratic longevity risk. New insights into the impact of systematic longevity risk, loadings for guarantee products and the potential impact of insurer insolvency risk on an individual's optimal product portfolio are provided.

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<sup>1</sup>Systematic longevity risk arises from uncertain changes in population survival probabilities that apply to all individuals to a greater or lesser extent.

Multi-period simulation is then used to assess a broader range of realistic individual portfolio strategies. The multi-period model incorporates stochastic mortality and inflation. Investment risk is not included in order to focus on products that manage longevity risk rather than investment risk.<sup>2</sup>

Individual welfare is compared for different product portfolios motivated by the optimal insurance literature (see, e.g., Borch, 1960; Arrow, 1971, 1973; Raviv, 1979). Coinurance for longevity risk is represented in the portfolio by self-insurance, referred to as phased withdrawals, and deductibles by deferred annuities. Applying concepts from the optimal insurance literature is complicated by the effect of a bequest motive in the longevity case. We address this by including cases with and without bequests to highlight the impact on individual longevity risk management.

The paper consists of two main parts. Section 2 presents a discrete state model to derive the optimal longevity insurance strategy for an individual facing both idiosyncratic and systematic longevity risk subject to a budget constraint. The individual has access to a complete market of financial and insurance products that allows the individual to attain optimal consumption in current and future states. The products required to complete the market are a risk-free bond, a life annuity, a longevity bond, and a GSA plan. Products are priced using a contingent claims approach. Frictional costs and insolvency risk are introduced. An example is used to highlight the main insights of the model.

Section 3 presents a multi-period expected utility simulation analysis of a range of longevity insurance strategies developed from the insights of the two-period model and concepts from optimal insurance. The longevity insurance strategies include portfolios of life annuities, deferred annuities, inflation-indexed annuities, GSA's, and phased withdrawals. Expected utilities and certainty equivalent consumption are used for welfare comparisons. The market price for insurer annuity products is determined as the actuarially fair insurance premium plus loadings typically observed in annuity

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<sup>2</sup>Although only risk free investments are included, allowing for investment risk changes the relative weighting of phased withdrawals in the individual portfolios but not the main conclusions of the study.

markets. In practice loadings cover costs of guarantees, adverse selection and costs of capital from regulatory capital requirements. The stochastic evolution of mortality rates is based on a multivariate mortality model in Wills and Sherris (2010) designed to study the pricing and risk management of longevity risk. The market model developed by Ngai and Sherris (2011) is used to generate future stochastic inflation and economic scenarios. This model simulates gross domestic product, interest rates, stock prices, and inflation.

The results of the study show that for individuals with no bequest motive and with no product loadings, annuitization strategies including small GSA holdings are optimal under systematic longevity risk. Inflation indexed annuities dominate, and because life annuities insure both systematic and idiosyncratic longevity risk, GSA's have a limited role. With loadings on guaranteed life annuity products, GSA plans, which are mutual and non-guaranteed, become significantly more attractive for individuals in managing their post-retirement longevity risk, replacing even annuitization products with inflation guarantees. For individuals with a bequest motive, coinsurance strategies in the form of self annuitization (phased withdrawals) dominate. Holdings of GSA plans increase significantly where there are loadings on guarantee products typical of these products.

## 2 Optimal longevity insurance: a two-period model

We study the optimal transfer of idiosyncratic and systematic longevity risk and demonstrate the impact of loadings and insolvency risk for longevity products on optimal longevity insurance using a two-period expected utility model.<sup>3</sup> At the start of the period, an individual is endowed with an initial wealth of  $W_0$  (his retirement savings). He chooses consumption  $C_0$  and a portfolio of financial and insurance products to obtain optimal second period consumption  $C_1$  in future states. Uncertainty at the end

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<sup>3</sup>Two-period models have been employed, for example, by Brown (2003), Valdez *et al.* (2006), and Schulze and Post (2010) to study the demand for annuities and for group self-annuitization funds.

of the period arises from both idiosyncratic and systematic longevity risk, introduced in Section 2.1.

The individual has access to a risk-free investment, a life annuity, a longevity bond, and a GSA fund. These products complete the market and are introduced in Section 2.3. Their prices are derived using the state-contingent claims approach outlined in Section 2.2. These products allow the individual to achieve the optimal consumption pattern based on his preferences.

Section 2.4 studies the optimal consumption problem when there are no frictional costs and products are priced at fair market prices. In Section 2.5 the market is no longer complete as frictional costs are introduced. Products issued by intermediaries (i.e., the life annuity) include loadings for guarantees and adverse selection. To allow for solvency and costs of capital, the life annuity is then assumed to have a (small) probability of not paying off fully when the population survival rate is high.

Section 2.6 uses a numerical example to demonstrate the insights of the model for optimal individual longevity risk management as well as the practical implications. The results are used as a basis for the product portfolios assessed in the multi-period simulations.

## **2.1 Idiosyncratic versus systematic longevity risk**

Systematic longevity risk is the risk that arises from shocks to population-level mortality rates that apply to all individuals to a greater or lesser extent, whereas idiosyncratic longevity risk is uncertainty in individual survival given the population mortality rates. The model includes systematic and idiosyncratic mortality risk using four different states at the end of the period determined by the random population survival rate and the survival status of the individual given the mortality rate. The states are denoted by  $(h, a)$  for a high population survival rate with the individual alive,  $(l, a)$  for a low population survival rate with the individual alive,  $(h, d)$  for a high population

survival rate with the individual dead, and  $(l, d)$  for a low population survival rate with the individual dead.

The probability that the population survival rate at the end of the period is high is denoted by  $\pi(h)$ ; a low population survival outcome occurs with probability  $\pi(l) = 1 - \pi(h)$ . The individual's survival outcome depends on the population survival rate. For example, the probability that the population survival rate is high and the individual is alive is given by  $\pi(h)\pi(a|h)$ . Table 1 summarizes the possible survival outcomes and the corresponding probabilities.

State	Population	Individual	Probability
$(h, a)$	high	alive	$\pi(h)\pi(a h)$
$(l, a)$	low	alive	$\pi(l)\pi(a l)$
$(h, d)$	high	dead	$\pi(h)\pi(d h)$
$(l, d)$	low	dead	$\pi(l)\pi(d l)$

Table 1: Summary of notation for states and survival probabilities in the two-period complete market model.

## 2.2 Contingent claims

A state-contingent pricing approach is used. We consider a complete market and assume that for each of the four states a contingent claim is available that pays off 1 in the state and 0 in all other states. The prices of these contingent claims are denoted by  $p^c(h, a)$  for the contingent claim paying 1 in the  $(h, a)$  state and zero otherwise. Similarly for the other states the state-contingent prices are  $p^c(l, a)$ ,  $p^c(h, d)$ , and  $p^c(l, d)$ , respectively. All products are portfolios of these contingent claims in the model.

## 2.3 Longevity insurance products

The state-contingent claims are used to price financial and insurance products. Table 2 summarizes the pay-offs for the following products in the individual's portfolio:

- A risk-free bond that pays 1 in each of the states with price

$$p_b = p^c(h, a) + p^c(l, a) + p^c(h, d) + p^c(l, d)$$

so that  $\frac{1}{p_b} = 1 + r$  where  $r$  is the risk-free interest rate.

- A life annuity that pays 1 if the individual is alive with price

$$p_a = p^c(h, a) + p^c(l, a).$$

- A longevity bond that pays 1 if survival probabilities are high with price

$$p_l = p^c(h, a) + p^c(h, d).$$

- A group self-annuitization (GSA) overlay contract that pays 1 if the individual is alive and survival probabilities are low with price

$$p_g = p^c(l, a).$$

This GSA overlay contract, when combined with the life annuity, is a standard GSA contract as described by Valdez *et al.* (2006) that pays off in both states in which the individual is alive, with lower payoffs in the high population survival-state  $(h, a)$  than in the low population survival-state  $(l, a)$ .

The longevity bond and the GSA overlay allow the individual to manage systematic longevity risk, the life annuity insures against both systematic and idiosyncratic longevity risk, and the risk-free bond provides a bequest. The individual selects a portfolio of these products to optimize consumption in the future uncertain states. Individuals can both purchase and issue these products subject to a budget constraint.



State	Risk-free bond	Annuity	Longevity bond	GSA
$(h, a)$	1	1	1	0
$(l, a)$	1	1	0	1
$(h, d)$	1	0	1	0
$(l, d)$	1	0	0	0

Table 2: Longevity product payoffs at time  $t = 1$ . These products are portfolios of the state contingent securities.

## 2.4 Optimal longevity insurance: the complete market case without frictional costs or insolvency risk

In the complete market without frictional costs or insolvency risk, the optimal longevity insurance portfolio that finances the optimal consumption pattern is determined using the contingent claims introduced in Section 2.2. Two cases are studied: Section 2.4.1 considers an individual with no bequest motive, whereas Section 2.4.2 allows for a bequest motive. The optimal financial and insurance product portfolio is then determined that replicates the optimal contingent claims portfolio.

### 2.4.1 Optimal consumption without bequest motive

An individual with initial wealth  $W_0$  (i.e., his retirement savings) determines his optimal consumption by maximizing expected utility over future uncertain states. The individual faces a budget constraint so that only wealth,  $(W_0 - C_0)$ , after initial consumption of  $C_0$ , can be used to finance contingent claims to consumption in each of the future states. Consumption at the end of the period  $C_1$  is uncertain. The consumption in each of the future uncertain states is denoted by  $C(h, a)$ ,  $C(l, a)$ ,  $C(h, d)$  and  $C(l, d)$ , respectively. The utility function is assumed to be additive and separable so that

$$U(C_0, C_1) = u(C_0) + \beta E[u(C_1)],$$

where  $\beta$  is the time preference parameter.

If the individual has no bequest motive then the optimal consumption in the dead state

is zero ( $C(h, d) = C(l, d) = 0$ ). There is no utility in the dead state in this case even if there is an unintended bequest. For the no bequest case, the utility function in the dead state is  $u(C) = 0$ . The optimal consumption problem for the individual is

$$\max_{C_0, C(h,a), C(l,a)} u(C_0) + \beta [\pi(h)\pi(a|h)u(C(h, a)) + \pi(l)\pi(a|l)u(C(l, a))]$$

subject to the budget constraint

$$W_0 - C_0 = p^c(h, a)C(h, a) + p^c(l, a)C(l, a)$$

The first-order conditions are used to derive the marginal rates of substitution between consumption at the start of the period and at the end of the period at the optimal consumption:

$$\begin{aligned} MRS_{C_0, C(h,a)} &= \frac{\frac{\partial u(C_0)}{\partial C_0}}{\frac{\partial u(C(h,a))}{\partial C(h,a)}} = \frac{\beta\pi(h)\pi(a|h)}{p^c(h, a)} \\ MRS_{C_0, C(l,a)} &= \frac{\frac{\partial u(C_0)}{\partial C_0}}{\frac{\partial u(C(l,a))}{\partial C(l,a)}} = \frac{\beta\pi(l)\pi(a|l)}{p^c(l, a)}. \end{aligned}$$

The marginal rate of substitution between consumption in the states where the individual is alive is:

$$MRS_{C(h,a), C(l,a)} = \frac{\frac{\partial u(C(h,a))}{\partial C(h,a)}}{\frac{\partial u(C(l,a))}{\partial C(l,a)}} = \frac{\frac{p^c(h,a)}{\pi(h)\pi(a|h)}}{\frac{p^c(l,a)}{\pi(l)\pi(a|l)}}.$$

This is the ratio of the state-contingent prices divided by the state probabilities. The consumption trade-off between states is determined by the price of transferring consumption between the states. The first-order conditions determine the individual's optimal consumption pattern  $C_{no\ bequest}^* = (C_0^*, C^*(h, a), C^*(l, a), 0, 0)$ .

### 2.4.2 Optimal consumption with bequest motives

Now consider an individual with a bequest motive. The individual derives utility from consumption in the dead state and to reflect the lower utility in this state the utility function is scaled by a factor  $k$  with  $0 < k < \infty$  (see, e.g., Campbell, 2008). The individual now faces the optimization problem:

$$\max_{C_0, C(h,a), C(l,a), C(h,d), C(l,d)} \left\{ u(C_0) + \beta \left[ \begin{array}{l} \pi(h)\pi(a|h)u(C(h,a)) \\ +\pi(l)\pi(a|l)u(C(l,a)) \\ +\pi(h)\pi(d|h)k u(C(h,d)) \\ +\pi(l)\pi(d|l)k u(C(l,d)) \end{array} \right] \right\}$$

subject to the budget constraint

$$W_0 - C_0 = p^c(h,a)C(h,a) + p^c(l,a)C(l,a) + p^c(h,d)C(h,d) + p^c(l,d)C(l,d).$$

As before, the first-order conditions can be rearranged to derive the marginal rates of substitution between consumption at the beginning of the period,  $t = 0$ , and at the end of the period,  $t = 1$ , and between the four different states at  $t = 1$ . These determine the optimal consumption pattern denoted by  $C_{bequest}^* = (C_0^*, C^*(h,a), C^*(l,a), C^*(h,d), C^*(l,d))$  for the individual with a bequest motive.

### 2.4.3 Optimal longevity insurance

The optimal consumption patterns  $C_{no\ bequest}^*$  and  $C_{bequest}^*$  are used to determine the optimal product portfolio  $\alpha^* = (\alpha_b^*, \alpha_a^*, \alpha_l^*, \alpha_g^*)$ , where the  $\alpha_i^*$ ,  $i = b, a, l, g$  denote the number of units the individual buys from the risk-free bond, the life annuity, the longevity bond, and the GSA overlay contract introduced in Section 2.3.

The optimal product portfolio  $\alpha^*$  allows the individual to finance the optimal con-

sumption pattern  $C^*$ . This portfolio is the solution to the system of equations:

$$\text{State}(h, a) : C^*(h, a) = \alpha_b + \alpha_a + \alpha_l$$

$$\text{State}(l, a) : C^*(l, a) = \alpha_b + \alpha_a + \alpha_g$$

$$\text{State}(h, d) : C^*(h, d) = \alpha_b + \alpha_l$$

$$\text{State}(l, d) : C^*(l, d) = \alpha_b$$

$$\text{Budget constraint} : W_0 - C_0 = p_b\alpha_b + p_a\alpha_a + p_l\alpha_l + p_g\alpha_g$$

The first four equations are determined from the product payoffs summarized in Table 2. The last equation is the budget constraint in terms of product prices and units in the portfolio.

## 2.5 Optimal longevity insurance: the case of frictional costs and insolvency risk

We now extend the model to include frictional costs in the form of loadings for guarantee products and insolvency risk. These impact the demand for the life annuity and the extent of coinsurance in the optimal product portfolio. Frictional costs and insolvency risk are introduced separately. In both cases we assume that the individual has a bequest motive to provide for an amount of wealth in the event of death at the end of the period.

### 2.5.1 Frictional costs

The life annuity now includes a loading so the price for the life annuity becomes the complete market price  $p_a$  plus a loading  $\phi$ :

$$p_{a,loading} = (1 + \phi)[p^c(h, a) + p^c(l, a)].$$

The individual optimizes the same expected utility as in the bequest case with no loadings and is subject to the same product cash flow constraints. However the individual now faces the following budget constraint:

$$W_0 - C_0 = p_b \alpha_b + p_{a,loading} \alpha_a + p_l \alpha_l + p_g \alpha_g$$

The individual maximizes his expected utility by choosing the consumption pattern  $C$  and the product portfolio  $\alpha$  given these product market prices. The resulting optimal consumption pattern is denoted by  $C_{loading}^* = (C_0^*, C^*(h, a), C^*(l, a), C^*(h, d), C^*(l, d))$ .

### 2.5.2 Insolvency risk

To allow for insolvency risk it is assumed that the life annuity pays only a proportion of the promised annuity payment in the event that survival probabilities are high and the full amount when survival probabilities are low. The probability that losses will not be fully covered is denoted by  $\pi_{insolvency}$  and the price for the annuity is given by:

$$p_{a,insolvency} = (1 - \pi_{insolvency}) p^c(h, a) + p^c(l, a).$$

The individual faces the same optimization problem as previously but with budget constraint:

$$W_0 - C_0 = p_b \alpha_b + p_{a,insolvency} \alpha_a + p_l \alpha_l + p_g \alpha_g$$

The individual maximizes his expected utility by choosing the consumption pattern  $C$  and the product portfolio  $\alpha$  given the product market prices and the resulting optimal consumption pattern is denoted by  $C_{insolvency}^* = (C_0^*, C^*(h, a), C^*(l, a), C^*(h, d), C^*(l, d))$ .

## 2.6 Numerical example

In order to provide insight into the practical implications of the two-period theoretical model, a numerical example is used to determine optimal product portfolios for an individual. Risk premiums for products are included and their impact on the optimal product portfolios assessed. Sensitivity of the results to these assumptions is also analyzed.

The individual is risk-averse, with decreasing absolute risk aversion ( $u'(C) > 0$ ,  $u''(C) < 0$ ,  $u'''(C) > 0$ ). A standard utility function satisfying these assumptions is the power utility function:

$$u(C) = \frac{C^{1-\delta} - 1}{1-\delta},$$

where  $\delta$  denotes the individual's relative risk aversion. A moderate relative risk aversion of  $\delta = 2$  is assumed and the time preference parameter is set to  $\beta = 0.98$ . The individual's initial wealth is  $W_0 = 100$  although since our utility function is homothetic the level of wealth does not impact the conclusions.

The probability that the population survival rate is high at  $t = 1$  is  $\pi(h) = 0.6$  (thus,  $\pi(l) = 0.4$ ), and the conditional probabilities that the individual is alive is  $\pi(a|h) = 0.9$  given a high population survival rate, and  $\pi(a|l) = 0.8$  given a low population survival rate. Table 3 gives the resulting state probabilities  $\pi(h, a)$ ,  $\pi(l, a)$ ,  $\pi(h, d)$ , and  $\pi(l, d)$  and the state-contingent prices  $p^c(h, a)$ ,  $p^c(l, a)$ ,  $p^c(h, d)$ , and  $p^c(l, d)$ . State-contingent prices are chosen so they result in reasonable market prices of the financial and insurance products. Prices are varied later to assess the results' sensitivity to this assumption.

State	Probability $\pi$	Price $p^c$
$(h, a)$	0.540	0.547
$(l, a)$	0.320	0.314
$(h, d)$	0.060	0.050
$(l, d)$	0.080	0.060

Table 3: Numerical example illustrating the two period model: State probabilities and state-contingent prices.

Table 4 gives the risk-neutral prices, the market prices and the risk premiums of the financial and insurance products resulting from the state prices in Table 3. The market price for the risk-free bond is  $p_b = 0.971$ , with a risk-free interest rate of  $1 + r = \frac{1}{p_b} = 1.03$ . The risk premium is defined as the expected return from a product in excess of the risk-free rate ( $1+r$ ). The expected return is calculated as the expected payoff divided by the market price. The risk premium is zero for the risk-free bond and negative for the annuity, the longevity bond, and the GSA overlay contract. The individual pays a risk premium to transfer his longevity risk. The risk premium is largest in absolute value for the life annuity, which transfers both systematic and idiosyncratic longevity risk, second largest for the longevity bond, which transfers only systematic longevity risk, and small for the GSA overlay contract, which only transfers idiosyncratic longevity risk.

Product	Risk-neutral price	Market price	Risk premium	$\alpha_{no\ bequest}^*$	$\alpha_{bequest}^*$
Risk-free bond	0.971	0.971	0.000	0.000	23.315
Annuity	0.835	0.861	-0.031	53.110	29.547
Longevity bond	0.583	0.597	-0.025	0.000	-1.087
GSA overlay	0.311	0.314	-0.010	0.882	-0.166

Table 4: Numerical example illustrating the two period model: prices, risk premiums, and optimal product portfolios with and without bequest.

Table 4 also shows the optimal portfolios for the no bequest motive case and with a bequest motive case where the utility from consumption in the “dead” state is scaled by the factor  $k = 0.15$ . These are denoted respectively by  $\alpha_{no\ bequest}^*$  and  $\alpha_{bequest}^*$ . An individual with no bequest motive buys 53.110 units of the annuity and 0.882 units of the GSA overlay contract. An individual with a bequest motive buys much less units of the annuity, in this case 29.054, and holds 23.315 units of the risk-free bond to provide a bequest. He also issues small amounts of the longevity bond and the GSA overlay contract. The relative prices including risk premiums for the life annuity, longevity bond and the GSA determine the demand for these products.

In the no bequest case, systematic longevity risk has limited impact on the optimal strategy of full insurance. Demand for full annuitization is the dominant strategy

even in the presence of systematic risk. This occurs because life annuities insure both idiosyncratic and systematic longevity risk. For individuals with a bequest motive, it is optimal to include consumption in the dead state and this is provided by holding the risk-free bond. There is a significantly reduced demand for the life annuity. The risk-free bond is the equivalent of coinsurance in the two-period example. Portfolio strategies that include coinsurance are optimal in the case of a bequest, even when this utility is heavily discounted as in our example.

Systematic risk is mostly insured with the life annuity. The GSA overlay and the longevity bond holdings are relatively small. As will be shown, when loadings on the life annuity increase substantially, the relative cost of hedging longevity risk with the life annuity becomes high enough that the demand for the GSA overlay and the longevity bond increases. Individuals then prefer to bear the systematic longevity risk rather than pay the higher insurance premium in the life annuity.

### **2.6.1 Numerical example: Loadings and insolvency risk**

In this section the effect of loadings and insolvency risk is considered using the same state-contingent claims prices as in the previous section for an individual with a bequest motive.

A loading of  $\kappa = 0.03$  on the market price of the life annuity is included. This changes the risk premium for this product to  $-0.060$ . Table 5 shows that the demand for the life annuity is significantly reduced from 29.547 to 18.736 units compared to the bequest case without loadings in Table 4. The individual now buys 8.466 units of the longevity bond and 10.500 units of the GSA overlay contract to insure his systematic longevity risk. A relatively small increase in the effective risk premium for the life annuity that arises from product price loadings has a significant impact on the product mix in the optimal portfolio.

Insolvency risk for the annuity provider is introduced by assuming a probability that there will be annuity payoffs at the end of the period of  $\pi_{insolvency} = 0.95$ . The life



annuity price includes this and the risk premium is unchanged. As expected, a similar optimal portfolio results as in the bequest case without insolvency risk. The individual with a bequest motive buys slightly more units of the annuity amounting to 31.228, up from 29.054, since insolvency risk is fairly reflected in the annuity's market price. He also holds 23.309 units of the risk-free bond to provide a bequest, and, as before, issues small amounts of the longevity bond and the GSA overlay contract.

Product	Loading on the life annuity		Insolvency risk	
	Risk premium	$\alpha_{loading}^*$	Risk premium	$\alpha_{insolvency}^*$
Risk-free bond	0.000	23.234	0.000	23.309
Annuity	-0.060	18.736	-0.031	31.228
Longevity bond	-0.025	8.466	-0.025	-1.196
GSA overlay	-0.010	10.500	-0.010	-1.897

Table 5: Numerical example illustrating the two period model: risk premiums and optimal product portfolios for the cases of loadings or insolvency risk with bequest.

With a bequest motive and loadings on the price of the life annuity, demand for the life annuity is significantly reduced. Demand for the risk-free bond, representing coinsurance, remains almost unchanged. Life annuity demand is substituted with holdings in the longevity bond and GSA that incur no loadings. Individuals place more of their retirement wealth into coinsurance portfolio strategies by holding mutual products without guarantee loadings.

The results for the case in which the life annuity cash flows are discounted for expected insolvency risk are very similar to the complete market case with a bequest motive. The individual buys more units of the life annuity because the risk premium of the life annuity is slightly lower than in the complete market case due to the reduced expected payout in the  $(h, a)$  state.

## 2.6.2 Numerical example: Impact of the price for transferring systematic and idiosyncratic longevity risk

The case with loadings on the life annuity highlights the important role that market prices play in the optimal portfolio. The impact of the price for transferring system-

atic and idiosyncratic longevity risk on an individual's optimal longevity insurance strategy is now considered for an individual with a bequest motive.

To begin, consider the case where there is no risk premium for transferring either systematic or idiosyncratic longevity risk. The corresponding market prices  $p^c(h, a)$ ,  $p^c(l, a)$ ,  $p^c(h, d)$ , and  $p^c(l, d)$  for the contingent claims are then equal to the state probabilities  $\pi(h, a)$ ,  $\pi(l, a)$ ,  $\pi(h, d)$ , and  $\pi(l, d)$ . The market prices for the financial and insurance products are equal to the product's risk-neutral prices.

Table 6 shows the optimal product portfolio  $\alpha_{zero\ rp}^*$  for the case of no risk premiums. There are 20.126 units of the risk-free bond, 31.839 units of the life annuity, no holdings of the longevity bond or the GSA overlay contract. There is very little difference in this portfolio from the optimal portfolio  $\alpha_{bequest}^*$ . The life annuity is the main form of longevity insurance. Relative risk premiums then determine holdings of the longevity bond and the GSA.

For the case where only systematic longevity has a loading there is a zero loading on the GSA overlay contract. The life annuity loading is lower and the loading for the longevity bond is unchanged compared to the examples in the previous subsections. Table 6 shows that the optimal product portfolio  $\alpha_{systematic\ rp}^*$  contains similar holdings of the risk-free bond and the annuity as the optimal portfolio  $\alpha_{bequest}^*$ . A small number of units of the longevity bonds are issued and a small number of GSA overlay contract purchased.

The individual's demand for the annuity and for the risk-free bond is mainly determined by preferences, including the bequest motive, rather than the product loadings for systematic and idiosyncratic longevity risk. The demand for longevity bonds and GSA contracts is mainly determined by the relative risk premiums between these products and the life annuity. Larger loadings for the life annuity reduce demand substantially and increase relative demand for the longevity bond and the GSA.

Product	Zero risk premiums		Risk premium only for systematic risk	
	Risk premium	$\alpha_{zero\ rp}^*$	Risk premium	$\alpha_{systematic\ rp}^*$
Risk-free bond	0.000	20.126	0.000	22.834
Annuity	0.000	31.839	-0.027	29.832
Longevity bond	0.000	0.000	-0.025	-0.857
GSA overlay	0.000	0.000	0.000	0.247

Table 6: Numerical example illustrating the two period model: risk premiums and optimal product portfolios for different prices for transferring systematic and idiosyncratic longevity risk

### 3 Optimal longevity insurance: multi-period numerical analysis

Motivated by the two-period model results, a multi-period stochastic simulation model is used to assess and compare individual product portfolio strategies. Systematic and idiosyncratic longevity is included using a stochastic mortality model along with inflation risk. Individual welfare is assessed using expected utility. Different ages at retirement are assessed. Premiums reflect actuarial discounted expected cash flows and include loadings.

We consider a range of portfolios. We include recent product innovations in the form of GSA products in the portfolios. We also include an inflation linked life annuity to compare portfolios with different exposures to longevity and inflation risk. Since there are no longevity bonds on issue, we no longer include a longevity bond. We consider portfolios with full annuitization and various coinsurance strategies with phased withdrawal, life annuities and the GSA. We also assess and compare the impact of a bequest motive and product loadings on individual welfare. We use certainty equivalent consumption for this purpose.

In a multi-period longevity risk setting, the equivalent of the two-period risk-free bond is phased withdrawals. This is where an individual draws down from his savings to provide a high probability of not exhausting his savings during his future uncertain lifetime. Full insurance is provided by an inflation linked life annuity so that con-

sumption in real terms is fixed.

Two types of self-insurance are considered. One is horizontal self-insurance, or co-insurance, where the individual uses a fraction of his wealth to purchase a life annuity and self insures with a phased withdrawal for the remainder. The other form of self-insurance is vertical self-insurance, or a deductible, where the individual purchases a deferred life annuity and uses the remaining wealth in a phased withdrawal over the fixed period until the deferred annuity commences payment.

### **3.1 Product cash flows and pricing assumptions**

Portfolios of the following products are assessed:

- Fixed life annuities paying a fixed nominal amount per period as long as the individual is alive. The price of the annuity is the actuarial expected present value based on population mortality plus a loading. The insurer issuing the annuity has an annual risk of insolvency. Cash flows for the life annuity are determined by the wealth allocated to the product divided by the expected annuity factor from the mortality model at the initial age including loadings.
- Deferred annuities paying fixed periodic annuity payments upon survival to a future age. The price of the annuity is the actuarial expected present value based on population mortality plus a loading. The insurer issuing the annuity has an annual risk of insolvency. The product is similar to the fixed life annuity except that payments only commence at age 85 provided that the individual is alive. Deferred annuity cash flows are the wealth allocated to the product divided by the expected deferred annuity value including loadings.
- Inflation-indexed annuities paying variable annuity payments per period that are indexed to the rate of inflation. The price of the annuity is the actuarial expected present value based on population mortality allowing for expected inflation plus a loading. The insurer issuing the annuity has an annual risk of insolvency. In

real terms, indexed annuity cash flows are fixed.

- Group self-annuitization (GSA) where payments depend on the population mortality experience and it is assumed that the size of the pool is large enough that idiosyncratic risk does not impact the individual payments. Payments depend on population mortality allowing for systematic mortality changes. If an individual is alive, the payment is the original life annuity value at the annuity value for the previous period divided by a revised annuity factor using expected future mortality at the next age. Payments vary with expected future mortality rates. For every economic scenario the distribution of mortality rates is allowed for in computing the utility of the cash flows. There is no insolvency risk for the GSA since it is a mutual fund.
- Self-annuitization, or phased withdrawal, is the equivalent of self-insurance. An individual draws down a regular annual payment such that, allowing for interest earnings at the fixed interest rate, the probability of running out of wealth at the initial age is less than 5% based on the expected future mortality rates. The payment is the amount drawn down in each period and continues until an individual dies or runs out of funds if he lives too long. If death occurs the remaining balance is an unintended bequest.

Expenses are not explicitly included. Cost-related supply-side factors for guaranteed annuity products are included with a loading in the price of the life annuity, the deferred annuity, and the inflation-linked annuity. Where expected insolvency is included the insurer issuing those annuity products is assumed to become insolvent with probability  $p_d = 0.005$ <sup>4</sup> and future cash flows in the event of insolvency are reduced by 95%. Annuity cash flows are multiplied by an expected insolvency factor  $0.995 + 0.005 * 0.95$  consistent with the two-period model.

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<sup>4</sup>This ruin probability corresponds to the once in 200 years ruin probability targeted under Solvency II (CEIOPS, 2010).

### 3.2 Market model

The market model used is that in Ngai and Sherris (2011). The model generates consistent future scenarios for four financial and economic times series (log gross domestic product (GDP), log bond index, log stock price index, and log inflation index). In this model GDP and the stock price index are included as macroeconomic variables that influence interest rates and inflation.

Each of these economic and financial variables is modeled as a time series vector  $\mathbf{y}_t$  using an econometric model that captures long-run relationships between the variables and allows for volatility to vary over time using a regime-switching assumption. The model used is a cointegrating vector error correction model with regime switching (RS-VECM). The general form of a RS-VECM with lag of  $p$  is expressed as:

$$\Delta \mathbf{y}_t = \boldsymbol{\mu} + \sum_{i=1}^{p-1} \mathbf{A}_i \Delta \mathbf{y}_{t-i} + \mathbf{B} \mathbf{C} \mathbf{y}_{t-p} + \boldsymbol{\varepsilon}_t(\boldsymbol{\omega}_t)$$

where  $\mathbf{y}_t$  is a  $d$ -dimensional vector of the economic and financial series,  $\Delta \mathbf{y}_t = \mathbf{y}_t - \mathbf{y}_{t-1}$  is the first differenced series,  $\boldsymbol{\mu}$  is the mean vector of rates of change in the variables,  $\mathbf{A}$  is a  $d \times d$  parameter matrix of coefficients,  $\mathbf{B}$  and  $\mathbf{C}'$  are  $d \times r$  matrices of rank  $r$  capturing the cointegration (equilibrium) relationship between the variables, and  $\boldsymbol{\varepsilon}_t(\boldsymbol{\omega}_t)$  is a vector of regime-dependent multivariate normal random errors with covariances  $\boldsymbol{\Sigma}_{\omega_t}$ :

$$\boldsymbol{\varepsilon}_t(\boldsymbol{\omega}_t) \sim \mathcal{N}_d(\mathbf{0}, \boldsymbol{\Sigma}_{\omega_t})$$

The model assumes two regimes, representing a normal state and high-volatility state and the probabilities of switching between regimes are modeled using a Markov chain.

The parameter estimates are taken from Ngai and Sherris (2011) where the model was selected as the best fitting VAR model with regime switching and cointegration based on quarterly data from the Reserve Bank of Australia (RBA) over the period 1970 to 2009. Accumulated 90-Day Bank-Accepted-Bill Yields are used for the bond index, the ASX All Ordinaries is used for the stock price index, and the inflation index is

constructed based on the CPI.

The economic scenarios generate future inflation rates. Interest rates are assumed fixed at the long-run average interest rate in the economic scenario model in order to isolate the impact of longevity risk from the impact of investment risk.

Three thousand economic scenario paths are generated. For this number of simulations the standard error of the estimated expected utility was at a low enough level to allow accurate comparison and ranking of alternative strategies.

### 3.3 Mortality model

The mortality model used to simulate future population mortality is based on the model used in Wills and Sherris (2010). The model is a discrete time, discrete age model of a vector of mortality rates for ages  $x = x_1, \dots, x_N$  denoted by:

$$\underline{\mu}(t) = \begin{bmatrix} \mu(x_1, t) \\ \vdots \\ \mu(x_N, t) \end{bmatrix}.$$

The model is based on rates of change in mortality rates for cohorts. The rate of change in mortality rates is given by

$$\Delta_c \ln \mu(x, t) = \ln \mu(x + 1, t + 1) - \ln \mu(x, t) = \ln \left( \frac{\mu(x + 1, t + 1)}{\mu(x, t)} \right)$$

where  $\Delta_c$  indicates differencing in the cohort direction.

The model assumes that the rates of change in the mortality rates in the cohort direction have a mean that depends on the age in the form

$$\Delta_c \ln \mu(x, t) = (a(x) + b)\Delta t + \sigma \Delta Z_x.$$

where  $a$ ,  $b$  and  $\sigma$  are constants and  $Z_x$  contains standard normally distributed random

variates that are correlated across ages with covariance matrix  $\Sigma$ .

To simulate future randomness, the eigenvalues and eigenvectors of the standardized residuals are used. Define  $\underline{\theta} = [\theta_1, \dots, \theta_N]'$  as the ordered eigenvalues of  $\Sigma$ , with  $\theta_1 \geq \dots \geq \theta_N$ . The corresponding eigenvectors are given by the matrix  $V = [V_1, \dots, V_N]$  where

$$\begin{aligned}\Sigma V &= VT \\ \Sigma &= VTV',\end{aligned}$$

and  $T$  is the  $N \times N$  diagonal matrix with diagonal  $\underline{\theta}$  and  $(V\sqrt{T})$  is the Cholesky decomposition of  $\Sigma$ .

Mortality rates are simulated by generating an  $N$ -dimensional random vector  $\underline{\eta} \sim iid N(0, I)$ , where  $I$  is an  $N \times N$  unit diagonal matrix. A random vector  $\underline{v}$  is then generated using

$$\underline{v} = V\sqrt{T}\underline{\eta},$$

where  $\sqrt{T}$  is an  $N \times N$  diagonal matrix with diagonal  $\sqrt{\underline{\theta}} = [\sqrt{\theta_1}, \dots, \sqrt{\theta_N}]'$ . The vector  $\underline{v}$  is normally distributed with covariance matrix:

$$\begin{aligned}\Sigma_v &= (V\sqrt{T})I(V\sqrt{T})' \\ &= VTV' \\ &= \Sigma.\end{aligned}$$

The model was calibrated to Australian Population Mortality Data for ages 65-99 from 1971-2004 from the Human Mortality Database, University of California, Berkeley (USA), and the Max Planck Institute for Demographic Research (Germany). Parameter values were re-estimated for Australian data for ages 65 and older. Estimated parameter values used were  $a = -0.00225071$ ,  $b = 0.237739$  and  $\sigma = 0.098440$ .

One thousand mortality paths are generated for the simulations.



### 3.4 Expected utility model

Products are compared using expected utility of simulated product cash flows. The expected utility for a particular product portfolio is derived by averaging the utility of consumption in real terms for the portfolio cash flows over the simulated outcomes for the economic variables and survival probabilities (both systematic and idiosyncratic), with time preference discounting.

The utility assumption without a bequest motive is standard and of the form

$$E \left[ \sum_{t=0}^{\omega-x} {}_t p_x \beta^t u(C_t) \right].$$

where  ${}_t p_x$  is the probability that an individual aged  $x$  years survives another  $t$  years and the expectation is taken over all economic and survival scenarios.

The one-period utility function is

$$u(C) = \frac{C^{1-\delta} - 1}{1-\delta},$$

consistent with the two-period example, with  $\delta$  denoting the individual's relative risk aversion. As in the two-period example, a moderate relative risk aversion of  $\delta = 2$  is used, the time preference parameter is  $\beta = 0.98$ , and the utility the individual derives from leaving a bequest is scaled by the factor  $k = 0.15$ .

Denoting the  $n$  mortality scenarios using  $a_s$  and the  $m$  economic scenarios using  $e_s$ , the expected utility for each alternative portfolio, risk aversion parameter, initial age, and initial wealth is computed using

$$\bar{U} = \sum_{t=0}^{\omega-x} \sum_{a_s}^n \sum_{e_s}^m \beta^t [I_a(t) \times u(C(a_s, e_s, a), t) + I_d(t) \times k \times u(C(a_s, e_s, d), t)]$$

where  $I_a(t) = 1$  if the individual is alive at time  $t$  and  $I_d(t) = 1$  if the individual is dead at time  $t$ .  $C(a_s, e_s, a)$  is the total cash flows from the product portfolio if the individual is alive and  $C(a_s, e_s, d)$  is the total cash flows from the product portfolio if

the individual is dead. Cash flows in the dead state only occur for product portfolios that include phased withdrawals where there are cash flow in the dead state. For utility in the dead state, the utility is reduced by  $k = 0.15$  as for the two-period example.

Since the idiosyncratic mortality scenarios average to the expected probabilities of survival and death at future dates this becomes

$$\bar{U} = \sum_{t=0}^{\omega-x} \sum_{e_s}^m \beta^t [\Pr(a, t) \times u(C(a_s, e_s, a), t) + \Pr(d, t) \times k \times u(C(a_s, e_s, d), t)]$$

where  $\Pr(a, t)$  is the probability the individual will be alive at time  $t$  and  $\Pr(d, t)$  is the probability the individual will die at time  $t$ . These are estimated from the mortality model by averaging across the simulated paths.

The cash flows are

$$C(a_s, e_s, a) = \sum_i w_i c f_i(a_s, e_s, a)$$

where  $i$  indicates the product,  $w_i$  is the amount invested in product  $i$ ,  $c f_i(a_s, e_s, a)$  is the cash flow for product  $i$  in that scenario for the “alive” state. A similar computation is used for the cash flows,  $C(a_s, e_s, d)$ , for product  $i$  in the scenarios for the “dead” states.

### 3.5 Portfolios

Differing retirement wealths of \$75,000, \$150,000, \$350,000 and \$750,000 are shown in the base case. As expected, certainty equivalents scale linearly in initial wealth so results for differing wealths are only shown in the base case. Individuals are assumed to retire at either 65, 75 or 85 to assess the impact of deferring the retirement decision on the optimal product portfolio.

Table 7 summarizes the portfolio percentages used for the different product portfolios. Buying a deferred annuity can be considered as a form of vertical self-insurance, or insurance with a deductible, of longevity risk, whereas self-annuitization corresponds to horizontal self-insurance, or co-insurance. A GSA provides horizontal self-insurance,

or co-insurance, of systematic longevity risk.

The portfolios are as follows:

- Portfolio 1: an individual fully insures nominal cash flows by purchasing a life annuity with the full amount of retirement wealth.
- Portfolio 2: an individual fully insures real cash flows with a variable payment inflation-indexed life annuity.
- Portfolio 3: an individual only insures idiosyncratic risk and bears the systematic risk in a mutual GSA. Loadings are lower than for a life annuity or deferred annuity.
- Portfolio 4: an individual purchases no insurance and self-annuitizes the full amount of retirement savings with a phased withdrawal.
- Portfolio 5: an individual self-annuitizes 75% of retirement wealth and purchases a deferred annuity with the remaining retirement wealth. This is a combination of vertical and horizontal self-insurance of longevity risk.
- Portfolio 6: an individual uses 50% of wealth to purchase a life annuity and the remainder is used in self-annuitization. This is horizontal self-insurance or co-insurance.
- Portfolio 7: an individual uses 50% of wealth to purchase a deferred life annuity and the remainder invested in a GSA. This is a combination of vertical self-insurance with horizontal self-insurance of systematic mortality risk.
- Portfolio 8: an individual combines a life annuity with a mutual GSA along with self-annuitization. 35% of retirement wealth is in the life annuity, 35% is in the GSA, and 30% is in self-annuitization. This portfolio combines two forms of horizontal self-insurance.
- Portfolio 9: an individual pools idiosyncratic risk and bears the systematic risk using a GSA for a deferral period and a deferred annuity is purchased to cover the

old age systematic and idiosyncratic risk. 75% of retirement wealth is in the deferred annuity and 25% is in the GSA. Similar to portfolio 7, this is a combination of vertical self-insurance with horizontal self-insurance of systematic mortality risk.

- Portfolio 10: is a modification of portfolio 5. The individual invests 50% of his retirement wealth in a self-annuitization strategy with a phased withdrawal ending at age 84 and uses the remaining 50% of wealth to purchase a deferred annuity starting at age 85. This is a combination of vertical and horizontal self-insurance of longevity risk.

A portfolio consisting of only a deferred annuity is not included since an individual requires consumption in every period including those prior to the deferral period commencing. In all cases with deferred annuities, there is a product included that generates cash flows prior to commencement of the deferred annuity.

Portfolio	Life Annuity	Deferred Annuity	Indexed Ann.	GSA	Self-Ann.
1	100				
2			100		
3				100	
4					100
5		25			75
6	50				50
7		50		50	
8	35			35	30
9		75		25	
10		50			50

Table 7: Summary of product mixes as percentages of total wealth. Each row shows the product mix for that portfolio.

# 4 Results

## 4.1 Economic indicators, survival rates, and annuity values

This section provides details of the expected (mean) values, standard deviations, and confidence intervals for the 3000 simulated economic scenarios and the 1000 mortality scenarios. Figure 1 gives the histograms for the GDP growth rates, annual interest rates, share price returns, and inflation rates based on the historical data and compares them with histograms for the same variables based on the simulated data over a 10 year horizon. The simulated scenarios are seen to capture the main distributional features of the historical data.

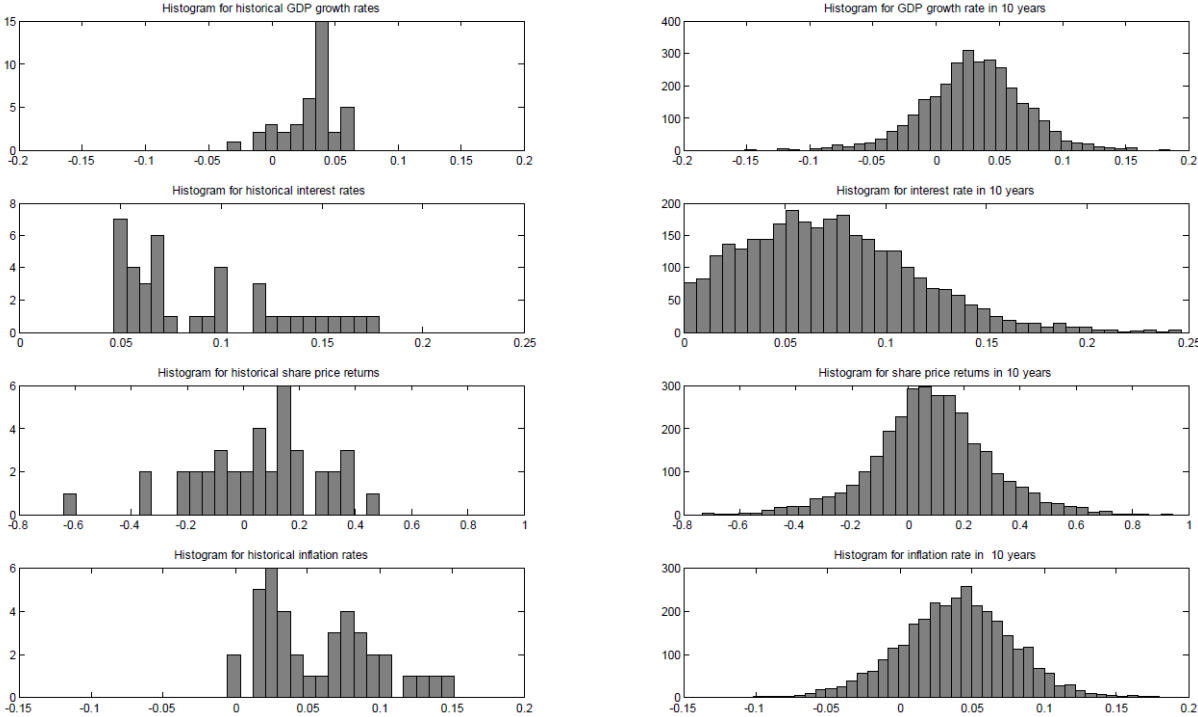


Figure 1: Historical data and simulated data after 10 years for GDP growth rates, annual interest rates, share price returns, and inflation rates.

Table 8 shows the expected multi-period survival probabilities for individuals aged 65-years in the first period. Figure 2 shows the corresponding expected survival curve. The expected probability of a 65-year old to survive to age 99 is 27.1% with a standard deviation of 18.2%. The model estimation was restricted to a maximum age of 99, so the probability of surviving to age 100 and beyond is zero. For the self-annuitization

strategy an individual withdraws a fixed annuity of income to the maximum age of 100. Earlier death results in a bequest.

Age	Survival probability ${}_t p_{65}$		Life annuity value		Inflation-indexed annuity value	
	Mean	Std. Dev.	Mean	Std. Dev.	Mean	Std. Dev.
65	1.000	0.000	10.275	0.474	14.234	1.017
70	0.935	0.003	9.461	0.685	12.700	1.271
75	0.845	0.015	8.592	0.923	11.160	1.516
80	0.728	0.044	7.658	1.133	9.603	1.672
85	0.591	0.092	6.619	1.196	7.979	1.609
90	0.454	0.140	5.325	1.038	6.129	1.277
95	0.339	0.171	3.433	0.558	3.738	0.627
99	0.271	0.182	0.934	0.000	0.963	0.000

Table 8: Simulated multi-period survival probabilities  ${}_t p_{65}$  and annuity values for the life annuity and the inflation-indexed life annuity for individuals aged 65 in the first period.

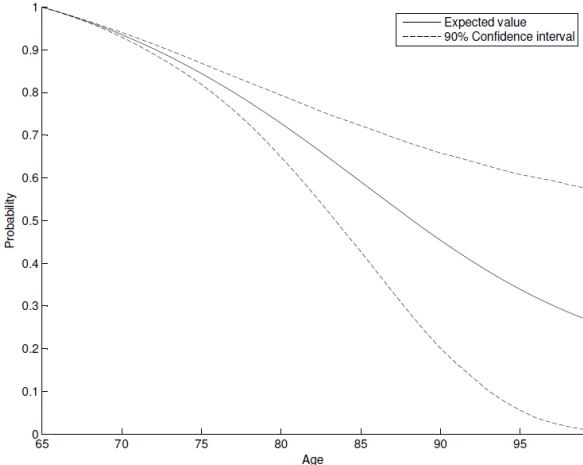


Figure 2: Survival curve for a 65-year old.

Table 8 also summarizes the simulated annuity values for the life annuity and inflation-indexed annuity for individuals aged 65 in the first period. Figure 3 and Figure 4 plot these values with confidence intervals showing the significance of systematic longevity risk. Because of indexation, life annuity values are lower than the inflation-indexed annuity values. For example, in the first period the annuity value for a 65-year old is 10.275 for the life annuity and 14.234 for the inflation-indexed life annuity since these are the values of an initial payment starting at 1 in both cases. Standard deviations increase at first with the increased uncertainty from the stochastic survival probabilities

and then diminish reflecting the effect of the smaller number of random cash flows in the annuity value.

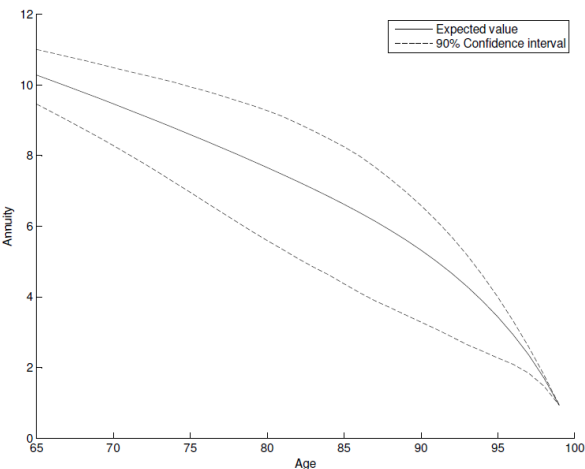


Figure 3: Annuity values for a 65-year old with confidence intervals.

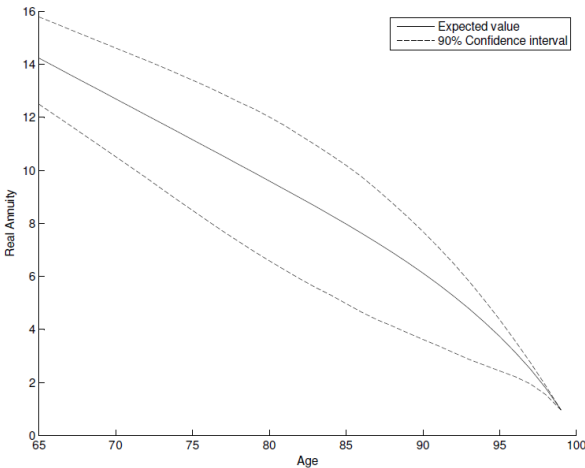


Figure 4: Indexed annuity values for a 65-year old with confidence intervals.

### 4.2 Portfolio comparison

The expected discounted utility values are estimated for each product portfolio over all 3000 economic scenarios and 1000 mortality scenarios. These simulated path numbers produced standard errors of the estimates that were small (effectively zero) compared to the average utility values.

The expected utility values are converted to certainty equivalent consumption levels. The certainty equivalent consumption *CEC* is the fixed yearly consumption level that

gives the same utility as the product portfolio (see, e.g., Stevens, 2010). The certainty equivalent consumption is calculated as:

$$E \left[ \sum_{t=0}^{\omega-x} {}_t p_x \beta^t \frac{(CEC)^{1-\delta} - 1}{1-\delta} \right] = \bar{U}.$$

where  $\bar{U}$  is the utility value calculated in equation (3.4),  $\beta = 0.98$  and  $\delta = 2$  as before, and  ${}_t p_x$  is the expected survival probability of an individual aged  $x$  at the beginning of the first period.

### The Base Case

In the base case, we consider a 65-year-old individual with no bequest motive and all the guarantee products have no loadings. Table 9 gives the certainty equivalent consumption and the preference-based ranking for each product portfolio for different wealth levels. Figure 5 plots the values confirming that for the utility function assumption these values scale linearly in wealth.

The preferred portfolio is Portfolio 2, which contains only inflation indexed annuities; the pure life annuity portfolio (Portfolio 1) is second best. The pure GSA strategy (Portfolio 3) ranks third, followed by the two portfolios that contain life annuities (Portfolio 8 and 6) and by Portfolio 5 that combines self-annuitization up to age 100 with the deferred annuity.

Less preferred portfolios are the the pure self-annuitization strategy (Portfolio 4) and the three remaining portfolios (Portfolios 10, 7, and 9). The last three portfolios all contain deferred annuities with portfolio 9 having the highest percentage of wealth (75%) invested in this product. These portfolios are unattractive because they have a low level of consumption in the early years before the cash flows from the deferred annuity contract commence.



Portfolio	\$75,000		\$150,000		\$350,000		\$750,000	
	CEC	Rank	CEC	Rank	CEC	Rank	CEC	Rank
1	4,252.09	(2)	8,504.20	(2)	19,843.21	(2)	42,520.48	(2)
2	5,267.93	(1)	10,535.84	(1)	24,583.61	(1)	52,679.72	(1)
3	3,999.35	(3)	7,998.68	(3)	18,663.70	(3)	39,993.34	(3)
4	3,416.47	(7)	6,832.93	(7)	15,943.58	(7)	34,165.01	(7)
5	3,587.09	(6)	7,174.16	(6)	16,739.73	(6)	35,870.61	(6)
6	3,834.55	(5)	7,669.12	(5)	17,894.67	(5)	38,345.68	(5)
7	3,032.56	(9)	6,065.12	(9)	14,151.96	(9)	30,325.56	(9)
8	3,955.21	(4)	7,910.41	(4)	18,457.66	(4)	39,551.96	(4)
9	1,562.23	(10)	3,124.45	(10)	7,290.41	(10)	15,622.29	(10)
10	3,057.67	(8)	6,115.36	(8)	14,269.15	(8)	30,576.87	(8)

Table 9: Certainty equivalent consumption and preference-based ranking for each product portfolio for different wealth levels. (Age = 65,  $\beta = 0.98$ ,  $\delta = 2$ , no bequest, no loadings)

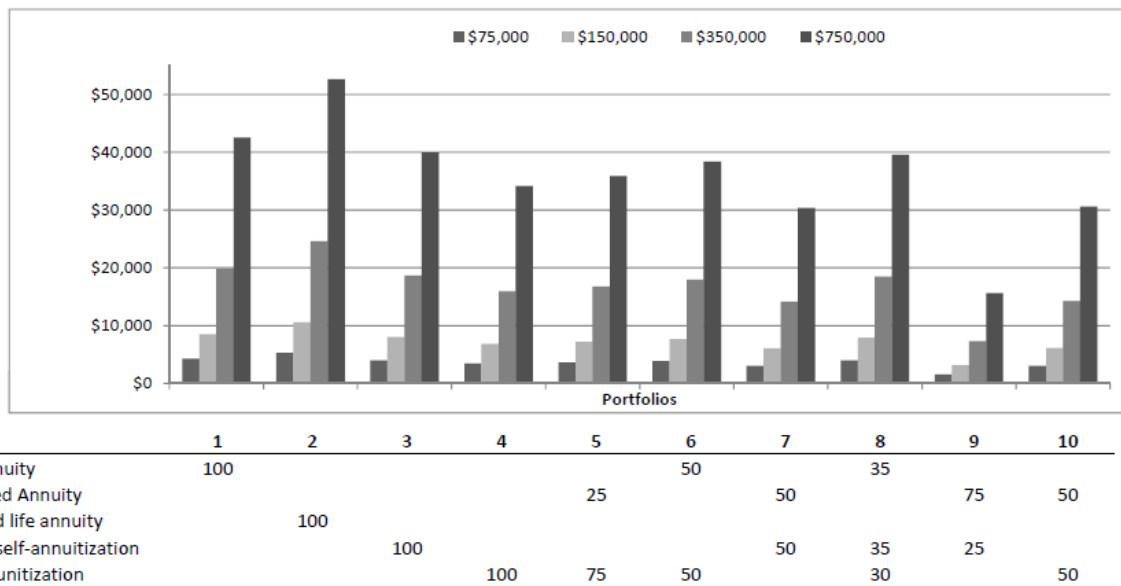


Figure 5: Certainty equivalent consumption for each product portfolio for different wealth levels. (Age = 65,  $\beta = 0.98$ ,  $\delta = 2$ , no bequest, no loadings)

### Different ages at retirement

Table 10 and Figure 6 compares the certainty equivalent consumption of individuals for different ages at retirement. The individuals all have an initial wealth of \$75,000, have no bequest motive, and the insurance products have no loadings.

Increasing the initial age to 75 slightly changes the preference-based ranking of the portfolios. Portfolio 10 now increases in ranking to 5th (up from 8th at age 65) and

Portfolio 7 has moved up from 9th to 7th ranking. Both of these portfolios contain deferred annuities. Portfolio 10 becomes more attractive because 50% of wealth is invested into a self-annuitization strategy that ends at age 84, after 10 years. This portfolio provides higher consumption in the early years before the deferred annuity starts.

At age 85, the preference-based ranking of the portfolio changes because for this age the deferred annuity starting at age 85 corresponds to an (immediate) life annuity. Thus, Portfolio 10 yields exactly the same cash flows, utility values, and certainty equivalent consumption level as the pure life annuity portfolio (Portfolio 1). Both portfolios rank second; the inflation-linked annuity portfolio (Portfolio 1) is still the preferred strategy. Portfolio 9, which was the least preferred portfolio with 75% of wealth invested in the deferred annuity, now ranks 4th. The least preferred portfolio at age 85 is the pure self-annuitization portfolio (Portfolio 4). The certainty equivalent consumption levels are higher since the same amount of wealth is consumed over a shorter period of time.

Deferring retirement makes portfolios with deferred annuities more attractive because consumption becomes smoother across time for the portfolios considered at the older retirement ages. The amount of deferred annuity purchased needs careful consideration since consumption needs to be spread across the deferral period as well as in the payment period of the deferred annuity. Decisions for individuals considering portfolios with deferred annuities are more complex because of the need to ensure smooth consumption across time.

### **Loadings on annuity products**

Table 11 and Figure 7 show the certainty equivalent consumption levels for the case of a 65-year-old individual without a bequest motive and with an initial wealth of \$75,000, but with varying levels of loading on annuity products. The base case of zero loading

Portfolio	Initial age 65		Initial age 75		Initial age 85	
	CEC	Rank	CEC	Rank	CEC	Rank
1	4,252.09	(2)	6,602.37	(2)	12,245.69	(2)
2	5,267.93	(1)	7,310.20	(1)	12,433.18	(1)
3	3,999.35	(3)	5,846.17	(3)	10,880.76	(6)
4	3,416.47	(7)	4,588.99	(9)	6,983.78	(10)
5	3,587.09	(6)	5,029.27	(8)	8,299.26	(9)
6	3,834.55	(5)	5,596.05	(6)	9,615.40	(8)
7	3,032.56	(9)	5,088.37	(7)	11,652.70	(5)
8	3,955.21	(4)	5,775.68	(4)	10,264.22	(7)
9	1,562.23	(10)	2,809.89	(10)	11,969.01	(4)
10	3,057.67	(8)	5,761.85	(5)	12,245.69	(3)

Table 10: Certainty equivalent consumption values and preference-based ranking for each product portfolio for different ages at retirement. ( $\beta = 0.98$ ,  $\delta = 2$ , wealth = \$75,000, no bequest, no loading)

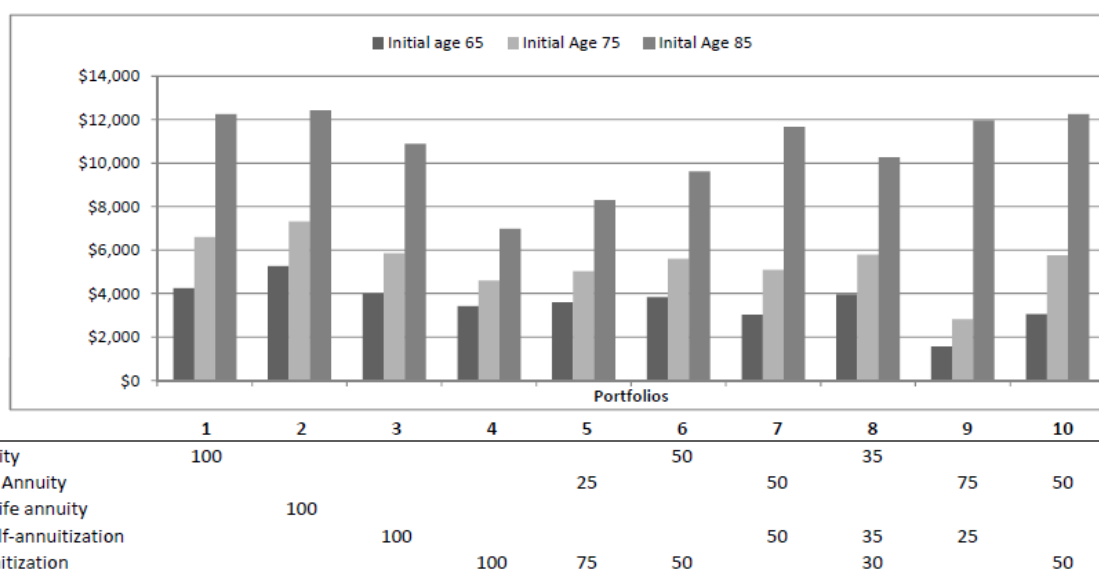


Figure 6: Certainty equivalent consumption values for each product portfolio for different ages at retirement. ( $\beta = 0.98$ ,  $\delta = 2$ , wealth = \$75,000, no bequest, no loading)

is compared with a 10% loading scenario and a 25% loading scenario.<sup>5</sup>

A loading of 10% changes only the relative ranking of two portfolios. The pure GSA portfolio (Portfolio 3) is now more attractive than the pure life annuity portfolio (Portfolio 2). Both portfolios rank 2nd and 3rd after the inflation-indexed annuity.

<sup>5</sup>Findings by Ganegoda and Bateman (2007) suggest that the loading on a nominal life annuity for a 65 year old male in the general population is around 24 per cent in the Australian market, which is more than double the previous estimates for Australian annuities reported in Doyle *et al.* (2004) using data for 2000.

The preference-based ranking of the portfolios changes significantly with a loading of 25% for annuity products. Now, the inflation-linked annuity (Portfolio 2), which was the preferred strategy previously, is dominated by the pure GSA strategy (Portfolio 3). Portfolio 8 which invests a large component in the GSA and the phased withdrawal, both without loadings, now ranks third. The pure life annuity portfolio (Portfolio 1) now ranks at 7th place.

Loadings in guaranteed annuity products are a significant factor influencing the demand for these products. The importance of mutual risk sharing arrangements such as GSA funds becomes much more significant in the presence of these loadings. This was demonstrated in the two-period model and confirmed in the multi-period simulation in a more realistic setting.

Portfolio	No loading		10% loading		25% loading	
	CEC	Rank	CEC	Rank	CEC	Rank
1	4,252.09	(2)	3,826.89	(3)	3,189.07	(7)
2	5,267.93	(1)	4,740.85	(1)	3,950.71	(2)
3	3,999.35	(3)	3,999.35	(2)	3,999.35	(1)
4	3,416.47	(7)	3,416.47	(7)	3,416.47	(5)
5	3,587.09	(6)	3,557.27	(6)	3,502.54	(4)
6	3,834.55	(5)	3,621.68	(5)	3,302.77	(6)
7	3,032.56	(9)	3,018.76	(9)	2,992.29	(9)
8	3,955.21	(4)	3,805.05	(4)	3,580.18	(3)
9	1,562.23	(10)	1,559.45	(10)	1,553.98	(10)
10	3,057.67	(8)	3,040.65	(8)	3,007.16	(8)

Table 11: Certainty equivalent consumption and preference-based ranking for each product portfolio for different loadings on annuity products. (Age = 65,  $\beta = 0.98$ ,  $\delta = 2$ , wealth = \$75,000, no bequest)

## Bequest

The two-period model showed the importance of the bequest motive on optimal portfolios to manage systematic longevity risk. A bequest motive is included for a 65-year-old individual with an initial wealth of \$75,000. Table 12 gives the resulting preference-based ranking of the ten product portfolios. For comparison, the second column of Table 12 shows the ranking for the case of an individual initially aged 65

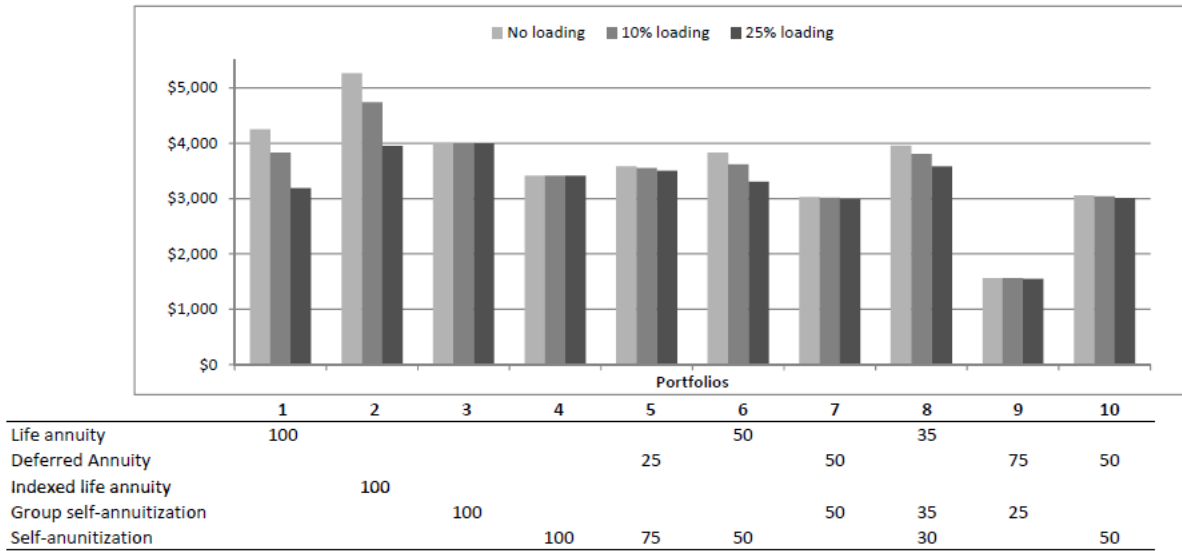


Figure 7: Certainty equivalent consumption for each product portfolio for different loadings on annuity products. (Age = 65,  $\beta = 0.98$ ,  $\delta = 2$ , wealth = \$75,000, no bequest)

without bequest motive without a loading, and the third and fourth columns give the results for a 10% and 25% loading on annuity products.

Introducing a bequest motive significantly impacts portfolios that contain phased withdrawal (Portfolios 4, 5, 6, and 8). These portfolios are more attractive because they include a bequest, which was unintended in the case where an individual had no bequest motive. The preference-based ordering of all portfolios is very different compared to the base case. The preferred strategy for an individual with bequest motive is the diversified coinsurance Portfolio 8, with 35% in the life annuity, 35% in the GSA, and 30% in self-annuitization. Portfolio 6, which combines the life annuity with 50% self-annuitization, is ranked second followed by Portfolio 5, which mixes the deferred annuity with 75% self-annuitization. The pure phased withdrawal strategy (Portfolio 4) ranks fourth, followed by Portfolio 10.

The results show that for an individual with a bequest motive, coinsurance portfolio strategies that include phased withdrawal and GSA's dominate full annuitization depending on the extent of product loadings.

Portfolio	Base Case	10% loading	25% loading	Bequest
1	(2)	(3)	(7)	(7)
2	(1)	(1)	(2)	(6)
3	(3)	(2)	(1)	(8)
4	(7)	(7)	(5)	(4)
5	(6)	(6)	(4)	(3)
6	(5)	(5)	(6)	(2)
7	(9)	(9)	(9)	(9)
8	(4)	(4)	(3)	(1)
9	(10)	(10)	(10)	(10)
10	(8)	(8)	(8)	(5)

Table 12: Preference-based ranking for the base case of an individual initially aged 65 without bequest motive with a retirement wealth of \$75,000 that is offered all insurance products without a loading and for different model variants.

## 5 Conclusions

The paper has assessed individual post-retirement longevity risk strategies for an individual facing both idiosyncratic and systematic longevity risk, inflation risk, and allows for bequests and product loadings. The individual holds a portfolio of financial and insurance products to optimize retirement consumption needs. Products include life annuities as well as recent innovations such as GSA's.

A theoretical framework based on state-contingent consumption and complete markets was presented and insight into the optimal strategy for an individual provided using a two-period model. The impact of a bequest motive was also assessed allowing for utility in the death state. The model showed that, with a bequest, strategies that include a risk free bond were optimal and that loadings on products that change their relative pricing have a significant effect on the optimal product portfolio required to manage both systematic and idiosyncratic longevity risk.

Multi-period simulation was used to assess and compare a broader range of retirement strategies with realistic simulations of economic variables and stochastic mortality with systematic and idiosyncratic risk. The product portfolios included traditional products such fixed life annuities, deferred annuities, inflation-indexed annuities, as well as group self-annuitization (GSA) plans, and phased withdrawals. The analysis

included product loadings and expected insolvency risk for annuity providers.

Individuals with no bequest motive, and with no product loadings, prefer annuitization strategies with small holdings of GSA plans under systematic longevity risk. With loadings on guaranteed life annuity products, GSA plans which are mutual and non-guaranteed, increase significantly in the preferred portfolios, replacing even annuitization products with inflation guarantees. For individuals with a bequest motive, portfolio strategies including self annuitization and GSA's dominate full annuitization.

Deferred annuity portfolios based on simple but reasonable retirement income portfolios are not preferred. They must be constructed carefully to include an appropriate level of consumption in the deferral period if they are to provide optimal consumption outcomes for individuals.

Importantly, under realistic assumptions, recent product innovations that manage systematic longevity risk (GSA plans) play a significant role in preferred portfolios. Product loadings for guarantee products can undermine the insurance welfare benefits of traditional annuity products.

## **6 Acknowledgement**

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