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Pooling functional disability and mortality in long-term care insurance and care annuities: a matrix approach for multi-state pools

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Pooling functional disability and mortality in long-term care insurance and care annuities: a matrix approach for multi-state pools

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Abstract

Mortality risk sharing pools such as pooled annuity funds and tontines provide an attractive and effective solution for managing longevity risk. They have been widely studied in the literature. However, such arrangements are not optimal for individuals in need of long-term care (LTC) insurance. Enhancing the design of pooled annuities and tontines factoring in LTC can aid in reducing the cost of LTC insurance. This paper presents a matrix-based approach for pooling mortality risk across heterogeneous individuals classified by functional disability states and chronic illness statuses. Based on multi-state models of functional disability and health statuses, we demonstrate how individuals with different health risks can share mortality risk in a pooled annuity design. A multi-state pool is formed by pooling annuitants vulnerable to longevity and LTC risks, determining the associated actuarially fair benefits based on individuals' health states. We provide a general structure for setting up a pooled annuity product that can be applied even for complex multi-state models. An extensive analysis is also carried out to illustrate our approach with numerical examples using US Health and Retirement Study (HRS) data. From the numerical illustrations, there is an increasing trend in the expected annuity benefits with higher upsides for individuals in poor health than those in good health, especially when systematic trends and uncertainty are considered in pricing. Smaller pool sizes and higher mortality credits among ill and disabled individuals due to higher death probabilities are the two main factors for the increased benefits in dependency.

Keywords: long-term care insurance, pooled annuity, multi-state models, functional disability, health status.

1 Introduction

Long-term care (LTC) risks and costs, along with longevity risk and its financing, have become an increasing concern in many countries in recent years. Longer life expectancies and low fertility rates have resulted in an ageing society with older adults who live longer, unhealthy lives and have a higher prevalence of disabilities as they age. Many OECD countries allocate a substantial share of the economy to LTC services and support. LTC expenditures in the United States account for 8.5% of total health expenditures ($135 billion in 2004) and about 1.2% of GDP, which has increased to 1.5% in 2018. The corresponding values in Australia and the Netherlands are 1.2% and 3.5% of GDP, respectively, as of 2018 (see Congressional Budget Office (2004), OECD (2020) and The Royal Commission Report (2020)).

Despite the substantial public costs, most of these expenditures are uninsured in many countries, mainly due to declining levels of informal care provided by families, insufficient public financing programs and a small private market (Brown and Finkelstein, 2007; Colombo et al., 2011). In the United States, only 4% of long-term care expenditure is covered by private insurance, with the remaining one-third paid out of pocket (Brown and Finkelstein, 2007). As societies age, there is increasing pressure to ensure the availability and

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affordability of long-term care (LTC) services for all those in need. This increases the demand for affordable, innovative private LTC insurance products required to meet the rising demand.

Several products have been proposed in literature to help develop the private LTC insurance market. LTC insurance policies are classified into four types according to Haberman and Pitacco (1999), Leung (2006) and Shao et al. (2017): fixed benefit policies sold to healthy individuals, fixed benefit policies sold to the elderly entering or already staying in LTC facilities, indemnity-based benefit policies, and policies that allow the insured to choose between fixed benefit and LTC service. The fixed benefit policy is the most common and widely used type of private LTC insurance. Fixed benefit LTC insurance can be purchased as a stand-alone policy or as a rider benefit in whole life insurance or life care annuities (Haberman and Pitacco, 1999).

A stand-alone policy pays out the predetermined benefit when the insured becomes functionally disabled. Some LTC insurance can also be combined with annuities, usually referred to as life care annuities (Murtaugh et al., 2001; Warshawsky, 2007; Brown and Warshawsky, 2013). A life care annuity reduces the adverse selection problem by pooling annuitants vulnerable to both longevity and LTC risks (Murtaugh et al., 2001). The risk pooling of the life care annuity provides a natural hedge and reduces insurance premiums (Murtaugh et al., 2001; Brown and Warshawsky, 2013). Specifically, for products with risk pooling, recent contributions in Hieber and Lucas (2022) and Chen et al. (2022) focusing on life-care or care-dependent tontines explain how pooling heterogeneous risks can be effective. Their analyses have shown that annuity payments improve due to additional mortality credits, especially for those in poorer health states.

In this paper, we consider risk pooling products, particularly a pooled annuity design or group self–annuitization structure proposed in Piggott et al. (2005). Mutual insurance plans such as pooled annuities have the potential to improve the annuity benefits through mortality and morbidity credits and have been shown to reduce the risk of adverse selection (Piggott et al., 2005; Valdez et al., 2006; Donnelly et al., 2013, 2014; Stamos, 2008; Qiao and Sherris, 2013; Hieber and Lucas, 2022; Chen et al., 2022). Our framework expands on the group self-annuitization structure introduced in Piggott et al. (2005) for sharing mortality risk to integrate sharing of functional disability and other health risks, such as chronic illnesses, in providing the LTC insurance.

Despite the extensive literature on pooling heterogeneous annuitants and evidence that it reduces adverse selection costs, the practical implications of mortality risk pooling have not been fully explored. Several studies including Sherris and Wei (2021) and Brown and Warshawsky (2013) demonstrate that mortality rates differ within functional disability states and according to chronic illness statuses, necessitating the need to pool different risks. However, most studies do not consider mortality risk sharing allowing for heterogeneous individuals, specifically individuals classified according to their functional disability states and chronic illness statuses to share mortality and morbidity risks. Also, the existing mortality sharing arrangements do not incorporate systematic trends and uncertainty in these risks. For instance, recent studies Hieber and Lucas (2022) and (Chen et al., 2022) assume no systematic risks affect pool members. Furthermore, their disability models tend to ignore recovery probability which is not negligible. The findings of Sherris and Wei (2021) motivate us to investigate the impact of pooling mortality, disability and health risks in a pooled annuity design.

Against this background, we present a framework for setting multi-state pools for heterogeneous individuals. A multi-state pool is formed by pooling individuals with different health risks and hence mortality risks and calculating annuity benefits for individuals in different states. We use a multi-state model to classify individuals according to their functional disability states and health statuses. We consider a three-state functional disability model and a five-state functional disability and health status model to classify individuals according to functional disability status only and functional disability and health status, respectively and examine the impact of this classification on pooled annuity payments. These models have been studied in Fu et al. (2021) and Sherris and Wei (2021), respectively, to simulate transition rates for calculating healthy (life) expectancy for the elderly.

This paper devises a framework on how heterogeneous individuals would share mortality risk in an actuarial fairly manner and how this impacts the annuity benefits depending on the individual's health status. The transition rates between different states are assumed to follow a multi-state latent factor intensity model proposed in Li et al. (2017), which includes systematic trends and uncertainties. This approach has been proposed, estimated, and applied in multi-state modelling in Fu et al. (2021) and Sherris and Wei (2021). As most sharing arrangements ignore the time trends and systematic uncertainties in functional disability and health dynamics, including the latent process with trend enables us to quantify the risk associated with future disability, chronic illness, and mortality rates and their impact on the pooled annuity benefits.

In estimating the transition rates, we consider a static or no-frailty model, a trend or no-frailty with a linear time trend model, and a frailty model to emphasize the importance of pooling different risks in the presence of systematic trends and uncertainties. For computing fund values and (discounted) pooled annuity payouts, we use a matrix-oriented technique recently proposed in Bladt et al. (2020) to derive Thiele-type theorems for calculating premiums and reserves in Markov chain models of life insurance.

We make two contributions to the literature. First, we present a theoretical framework for sharing mortality risk across multiple states, an arrangement we refer to as multi-state pooling. We use a matrix approach that provides a general structure for setting up a pooled annuity product, even for complex multi-state models. In previous studies such as Piggott et al. (2005) and Qiao and Sherris (2013), mortality risk pooling is based on the two-state (alive-dead) framework. We therefore extend the existing sharing arrangements to multiple states. Second, as a practical contribution, we propose a pooled health care annuity; a product which combines pooled life annuity and LTC insurance. Depending on an individual's health status or functional disability condition, the pooled health care annuity product adds to the need for long-term care coverage for an ageing population. It is more appealing to people with poor health and disabilities than a standard pooled annuity product, reducing adverse selection problems (Valdez et al., 2006). In addition, our design extends to the five-state framework rather than just a three-state model (as proposed in recent studies Hieber and Lucas (2022) and Chen et al. (2022), and includes time trends and systematic uncertainty. Pooling is made more effective when we allow for uncertainties with systematic improvements in mortality and morbidity rates (see discussions in Qiao and Sherris (2013) and Chen et al. (2020) for systematic mortality risk models applied to tontines and pooled annuities).

Results show an increasing trend in the expected annuity benefits with higher upsides for individuals in poor health (disabled and/or ill participants), especially when systematic trends and uncertainty are factored in pricing. Participants with functional disabilities and in ill states receive increased benefits due to the group's smaller pool size over long periods and higher mortality credit as a result of higher death probabilities. Ignoring systematic trends and uncertainty in pricing reduces both groups' annuity benefits at older, and the impact is much more significant for participants in good health due to the systematic nature of mortality risk among healthy participants. In addition, healthy participants always receive slightly below initial specified benefits even when considering the trends and uncertainty in the annuity factors, as they receive lower mortality and morbidity credits compared to other groups. Changes in pool size are another main factor affecting the pooled annuity benefits, where the variation in the benefit payments at future ages is very wide for smaller pool sizes.

The rest of this paper is organized as follows. Section 2 presents the modelling framework for multi-state pools, describing analytics of mortality risk pooling across multiple health states, the derivations of the annuity payouts based on the individual health status and the multi-state models' estimation procedures. Section 3 presents and discusses the main results using two multi-state models, a three-state functional disability model and a five-state functional disability and health status model, compared with the standard life care annuity. Concluding remarks are presented in Section 4. Appendix A presents the proof of the general framework.

2 Multi-state Pooling Framework

This section demonstrates how different risks can be shared in a pooled annuity design. We start with the two-state model, a simple multi-state framework where individuals share only mortality risk. We then derive a generalized framework that may be used for any multi-state model. The analysis shows how the proposed framework works for three and five-state models. The three-state model represents the functional disability model, illustrating how mortality and functional disability risks can be shared in a pooled annuity design. The five-state model, on the other hand, represents the functional disability and health status model to describe how individuals can share mortality, functional disability and chronic illness risks in a similar context. We focus on presenting the structure of actuarially fair pooled annuity benefits and deriving the annuity payouts from a multi-state process.

2.1 A two-state model pooling structure

A pooling arrangement in the two-state framework, such as the alive-dead model, is similar to the standard pooled annuity plan, where individuals share both systematic and idiosyncratic mortality risks. This structure is also considered in Piggott et al. (2005), but not in a multi-state context. We focus on deriving the pooled annuity payouts from a multi-state process.

Figure 1 illustrates the two-state alive-dead model. An individual is in one of two states at given time t: alive (A) or dead (D), and can only make one transition from State A to D. We adopt a discrete-time setting with each interval being of one-year length. Annuity benefits are paid while the individual is alive and cease when the individual dies. The payouts are adjusted annually based on the pool's realized mortality experience and investment performance.

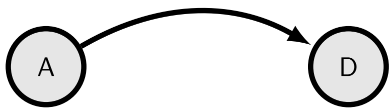


Figure 1: A two-state alive-dead model.

We first define two types of one year transition probabilities related to an active pool participant aged x:

- p_x: the probability that a person aged x will survive to age $x+1$;
- q_x: the probability that a person aged x will die before reaching age $x+1$.

Following the pioneering design in Piggott et al. (2005), at initial time, a pool consisting of l_x^* retirees aged x decides on the amount they expect to receive periodically in the future, B_0. This forms an initial total fund value defined as

$$F_0 = B_0 l_x^* \ddot{a}_x, \tag{2.1}$$

where F_0 is the pool fund value at time 0 and $\ddot{a}_x$ is the expected present value of a life annuity-due that pays 1 at the start of each period for an individual aged x calculated as

$$\ddot{a}_x = \sum_{t=0}^{\omega-x} \upsilon^t \, {}_tp_x, \tag{2.2}$$

with $\upsilon = \frac{1}{1+r}$ as the annual discount factor, r represents the expected interest rate assumed for pricing, ω is the maximum possible age and ${}_tp_x$ is the probability that a person aged x will reach age $x+t$.

Assuming there are no new entrants in the next period, at time 1, the entire group's fund value after interest earnings becomes

$$F_1 = l_{x+1}^* \left((p_x^*)^{-1} \left((F_0 - B_0 l_x^*)/l_x^* \right) \times (1+R_1) \right), \tag{2.3}$$

where R_1 is the deterministic financial return earned between the first interval, l_{x+1}^* is the actual number of participants in time 1 and $p_x^* = \frac{l_{x+1}^*}{l_x^*}$ is the actual survival probability in time 1. During the first interval, the individual's account value accrues to an amount of $((F_0 - B_0 l_x^*)/l_x^*) \times (1+R_1))$. This amount belongs to the actual number of survivors in time 1, l_{x+1}^*, and is distributed using the actual survivor probability.

If, for example, $l^*_{x+1} = l^*_x$, then $(p^*_x)^{-1} = 1$. The fund value for all participants in time 0 will be equal to that of all participants in time 1. In case of death of a participant, $(p^*_x)^{-1} > 1$, implying that the deceased participant's account value is lost and distributed to survivors in time 1 who will receive increased annuity payouts due to mortality credits.

The mortality credits on the individual's fund value can be represented as follows, first denote $F_{k,1}$ as the individual's fund value in time 1 defined from Equation (2.3) as,

$$F_{k,1} = \frac{F_1}{l^*_{x+1}} = (p^*_x)^{-1} \left((F_0 - B_0 l^*_x)/l^*_x \right) \times (1 + R_1), \tag{2.4}$$

Equation (2.4) can also be written as

$$F_{k,1} = (p^*_x)^{-1} F_{k,0} \times (1 + R_1), \tag{2.5}$$

where $F_{k,0}$ is the individual's fund value in time 0.

In terms of survival probabilities, we define $(p^*_x)^{-1} = \frac{l^*_x}{l^*_{x+1}}$, then $\theta_1 = \frac{l^*_x - l^*_{x+1}}{l^*_{x+1}}$ as mortality credits in time 1, so that $\frac{l^*_x}{l^*_{x+1}} = 1 + \theta_1$. Therefore, Equation (2.5) can be written as

$$F_{k,1} = F_{k,0} \times (1 + R_1)(1 + \theta_1), \tag{2.6}$$

where θ_1 is the mortality credits in time 1 and R_1 represents financial credits in time 1.

The pooled annuity payout at time 1 becomes

$$B_1 = \frac{F_1}{l^*_{x+1} \ddot{a}_{x+1}}, \tag{2.7}$$

where $\ddot{a}_{x+1}$ is the annuity factor at time 1.

Also by replacing F_1 in Equation (2.7) with Equation (2.3) and $\ddot{a}_{x+1} = (\ddot{a}_x - 1)(1 + r)/p_x$, Equation (2.7) can be written as follows

$$B_1 = B_0 \times \left(\frac{1 + R_1}{1 + r} \right) \times \left(\frac{p_x}{p^*_x} \right). \tag{2.8}$$

A recursive relationship between time t and $t + 1$ for the dynamics of the fund value and the pooled annuity payouts is presented as follows

$$F_{t+1} = l^*_{x+t+1} \left((p^*_{x+t})^{-1} \left((F_t - B_t l^*_{x+t})/l^*_{x+t} \right) \times (1 + R_t) \right), \tag{2.9}$$

and

$$B_{t+1} = B_t \times \left(\frac{1 + R_t}{1 + r} \right) \times \left(\frac{p_{x+t}}{p^*_{x+t}} \right), \tag{2.10}$$

where $\frac{1+R_t}{1+r}$ is an interest rate adjustment factor and $\frac{p_{x+t}}{p^*_{x+t}}$ is a mortality experience adjustment factor from period t to $t + 1$. Equation (2.10) is similar to the one derived in Piggott et al. (2005) for group self-annuity payouts. The critical aspect in the calculations shown above is that members who survive to time $t + 1$ benefit from those who died at time t, leaving them with significantly larger benefit payments (mortality credits). The mortality credits arise as members of the group die, and their fund value is divided among the survivors. The actual survival probability at time t, p^*_{x+t} in Equation (2.9) is used to distribute the mortality credits to the remaining pool participants. The next section presents a general framework for pooling mortality and health risks.

2.2 The general pooling framework

Assume that at any given time t, an individual is in any of the health states denoted by i. The individual's states are described by a multi-state model with $i = 1, 2, 3, \ldots, n, n+1$ states where $n+1$ is death (absorbing) state. There are $l_x^{i^*}$ individuals aged x in State i at the start of the period; annuity payments in that state B_t^i begin at the start of the period and continue until the maximum age ω or death, with transitions at the end of the period. We assume that there are no death benefit payments.

In the two-state model, annuitants receive the benefits in one state. Accordingly, for $n+1$-state model, the annuitants receive benefits in n states. The annuity factors are calculated using an $n \times n$ matrix of the expected one period transition probabilities. Let $\mathbf{P}_{x+t}$ be an $n \times n$ matrix of the estimated one period transition probabilities p_{x+t}^{ij} for a person aged $x+t$ at time t, where p_{x+t}^{ij} is the expected one period transition probability from State i to State j for an individual aged $x+t$ at time t. The matrix $\mathbf{P}_{x+t}$ is defined as

$$\mathbf{P}_{x+t} = \begin{bmatrix} p_{x+t}^{11} & p_{x+t}^{12} & \cdots & p_{x+t}^{1n} \\ p_{x+t}^{21} & p_{x+t}^{22} & \cdots & p_{x+t}^{2n} \\ \vdots & \vdots & & \vdots \\ p_{x+t}^{n1} & p_{x+t}^{n2} & \cdots & p_{x+t}^{nn} \end{bmatrix}, \tag{2.11}$$

where $x+t$ represents individual's age at time t, for $t = 0, 1, 2, \ldots T-1$, with T being the maximum period to receive the annuity benefits.

Also, let $\mathbf{A}_{x+t}$ be an $n \times n$ matrix of the annuity values represented as follows,

$$\mathbf{A}_{x+t} = \begin{bmatrix} a_{x+t}^{11} & a_{x+t}^{12} & \cdots & a_{x+t}^{1n} \\ a_{x+t}^{21} & \ddot{a}_{x+t}^{22} & \cdots & a_{x+t}^{2n} \\ \vdots & \vdots & & \vdots \\ a_{x+t}^{n1} & a_{x+t}^{n2} & \cdots & a_{x+t}^{nn} \end{bmatrix}. \tag{2.12}$$

We denote $\mathbf{A}_{x+t}$ as a matrix whose elements $a_{x+t}^{ij} = \sum_{y=0}^{\omega-(x+t)} \upsilon^y \ {}_y p_{x+t}^{ij}$ represent the expected present value of an annuity-due that pays a periodic payment of 1 at the beginning of the period for an individual aged $x+t$ in State ij alive at time t. Here, $\upsilon = (\frac{1}{1+r})$ is the discount factor and ${}_y p_{x+t}^{ij}$ is the expected transition probability from State i at age $x+t$ to State j at age $x+t+y$. The annuity value a_{x+t}^{ij} is obtained from one year transition probabilities matrices determined by a backward recursion formula as follows

$$\mathbf{A}_{x+t} = \mathbf{I} + \upsilon \mathbf{P}_{x+t} \mathbf{A}_{x+t+1}, \tag{2.13}$$

where $\mathbf{I}$ is an identity matrix of size n.

Similar to the GSA structure introduced in Piggott et al. (2005), the pool fund value at time t is defined by a matrix $\mathbf{F}_t$ which can be determined as

$$\mathbf{F}_t = \mathbf{L}_{x+t} \otimes (\mathbf{A}_{x+t} \mathbf{B}_t), \tag{2.14}$$

where $\mathbf{F}_t = \begin{bmatrix} F_t^1 \\ F_t^2 \\ \vdots \\ F_t^n \end{bmatrix}$ is an $n \times 1$ matrix of the pool fund value F_t^i for individuals in State i, $\mathbf{L}_{x+t} = \begin{bmatrix} l_{x+t}^{1^*} \\ l_{x+t}^{2^*} \\ \vdots \\ l_{x+t}^{n^*} \end{bmatrix}$

is an $n \times 1$ column vector of the actual number of participants in State i $l^{i^*}_{x+t}$ and $\mathbf{B}_t = \begin{bmatrix} B^1_t \\ B^2_t \\ \vdots \\ B^n_t \end{bmatrix}$ is an $n \times 1$ column vector of the annuity benefits B^i_t for individuals in State i, for $i = 1, 2, \ldots, n$ at any time t. In Equation (2.14), $\otimes$ is the Hadamard product for multiplying each row of the column matrix $\mathbf{L}_{x+t}$ to each row of the matrix $\mathbf{A}_{x+t}\mathbf{B}_t$, to compute the corresponding fund values for participants in State i.

We develop a general recursive formula between time t and $t+1$ for the dynamics of the fund value and the pooled annuity payouts presented for an $n+1$-state model. The following recursive relationship describes the dynamics of the pool fund value for the next period

$$\mathbf{F}_{t+1} = \mathbf{L}_{x+t+1} \otimes \left((\mathbf{P}^*_{x+t})^{-1} \left(\frac{\mathbf{F}_t - \mathbf{L}_{x+t} \otimes \mathbf{B}_t}{\mathbf{L}_{x+t}} \right) (1 + R_t) \right), \tag{2.15}$$

where

$$\mathbf{L}_{x+t+1} = \begin{bmatrix} l^{1^*}_{x+t+1} \\ l^{2^*}_{x+t+1} \\ \vdots \\ l^{n^*}_{x+t+1} \end{bmatrix} \tag{2.16}$$

is an $n \times 1$ matrix of the observed number of participants in State i over a one period $l^{i^*}_{x+t+1}$, and $\mathbf{P}^*_{x+t}$ is an $n \times n$ matrix of the observed one period transition probabilities $p^{ij^*}_{x+t}$.

The pooled annuity benefits in the next period are determined by

$$\mathbf{F}_{t+1} = \mathbf{L}_{x+t+1} \otimes (\mathbf{A}_{x+t+1}\mathbf{B}_{t+1}). \tag{2.17}$$

Based on Equation (2.15), the matrix $(\mathbf{P}^*_{x+t})^{-1}$ is multiplied to the individual's fund value after the interest earnings to determine the fund value required to provide the annuity benefits as the individual moves to different states. The pool fund value is then computed by element-wise multiplication of the individual's fund value in State i and the corresponding number of participants in that state. Hence, the first element-wise product determines the different pool fund values for individuals in various states, and the second element-wise product determines the benefits distributed for individuals in State i.

The actual number of participants in State i for the next period $l^{i^*}_{x+t+1}$, is calculated using the following relationship

$$l^{i^*}_{x+t+1} = l^{i^*}_{x+t} - \sum_{\substack{j=1 \\ j \neq i}}^{n} l^{ij^*}_{x+t+1} - l^{id^*}_{x+t+1} + \sum_{\substack{j=1 \\ j \neq i}}^{n} l^{ji^*}_{x+t+1}, \tag{2.18}$$

where $l^{i^*}_{x+t}$ is the realized number of participants in State i alive at time t, $l^{ij^*}_{x+t+1}$ is the realized number of participants alive transitioning out of State i to State j over one period, $l^{id^*}_{x+t+1}$ is the actual number of deceased participants in one period, and $l^{ji^*}_{x+t+1}$ is the number of participants alive transitioning into State i from State j over the period. Individuals in a pool have both idiosyncratic and systematic risks (Piggott et al., 2005). To incorporate idiosyncratic risk in numerical examples in Subsection 3.1, the actual number of individuals transitioning to and from State i between time t and $t+1$ is generated as a random draw from a Multinomial distribution as follows

$$l^{ij^*}_{x+t+1} \sim \text{Multi}\ (l^{i^*}_{x+t},\ p^{i}_{x+t}), \tag{2.19}$$

where p^i_{x+t} is the estimated one year transition probability for individuals in State i.

Based on Equation (2.18), we can then determine $p^{ij^*}_{x+t} = \frac{l^{ij*}_{x+t+1}}{l^{i*}_{x+t}}$ as the realized one period transition probability for an individuals aged $x+t$ at time t, hence $\mathbf{P}^*_{x+t}$ is an $n \times n$ matrix of the realized one period transition probabilities $p^{ij^*}_{x+t}$ defined as

$$\mathbf{P}^*_{x+t} = \begin{bmatrix} p^{11^*}_{x+t} & p^{12^*}_{x+t} & \dots p^{1n^*}_{x+t+1} \\ p^{21^*}_{x+t} & p^{22^*}_{x+t} & \dots p^{2n^*}_{x+t+1} \\ \vdots & \vdots & \vdots \\ p^{n1^*}_{x+t} & p^{n2^*}_{x+t} & \dots p^{nn^*}_{x+t} \end{bmatrix}. \tag{2.20}$$

If it happens that the realized investment earning rate R_t is the same as the expected rate r and the realized mortality as well as morbidity experience is the same as the expected experience that is $\mathbf{P}^*_{x+t} = \mathbf{P}_{x+t}$, then $\mathbf{B}_{t+1} = \mathbf{B}_t$. This implies that if financial and biometric risks behave as expected, the benefits will remain unchanged. A proof of this result is shown in Appendix A.

2.3 A three-state model pooling structure

Figure 2 illustrates the three-state health transition model commonly used to describe the state of an individual insured under a disability income policy. There are three states namely: Healthy (H), Functionally disabled (F), and Dead (D), with the following feasible transitions $H \to F$, $F \to H$, $H \to D$ and $F \to D$. Under this framework, it is possible to transfer from State F to H, that is, to recover from disability.

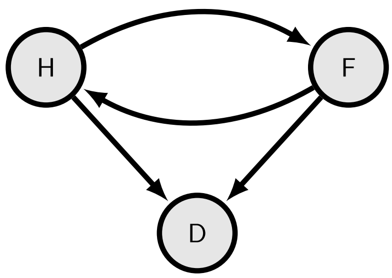


Figure 2: A three-state functional disability model allowing for recovery.

We define the following one year transition probabilities for an individual aged x:

- p^{hh}_x: the probability that a healthy person aged x will remain healthy at age $x+1$;
- p^{hf}_x: the probability that a healthy person aged x will become functionally disabled at age $x+1$;
- p^{hd}_x: the probability that a healthy person aged x will die before reaching age $x+1$;
- p^{ff}_x: the probability that a functionally disabled person aged x will remain functionally disabled at age $x+1$;
- p^{fh}_x: the probability that a functionally disabled person aged x will recover from disability at age $x+1$;
- p^{fd}_x: the probability that a functionally disabled person aged x will die before reaching age $x+1$.

We focus on LTC insurance which provides a pool participant with financial support, in particular, enhanced annuity benefits while they need nursing and medical care due to chronic or ongoing conditions such as functional disability. The annuity payouts for the LTC contracts are characterized by uplifts to basic life annuity benefits (Pitacco, 2014). The life care annuity versions in Murtaugh et al. (2001), Warshawsky (2007), Brown and Warshawsky (2013), Pitacco (2013) and Pla-Porcel et al. (2017) are typical examples for such products. The benefits are payable while the insured is healthy as a life annuity or functionally disabled as an LTC claim. The insurer takes all mortality and morbidity risks.

We propose a life care annuity embedded in a pooled annuity fund where all participants bear the risks by following a standard life care annuity design presented in Brown and Warshawsky (2013). As shown in Figure 2, the individual is in one of the three states at any given time. The annuity benefits are payable in two states: H and F.

Assume the standard life care annuity which pays B_0^h at the beginning of each period as long as the individual is healthy and $B_0^f = (1+c) \times B_0^h$ when the individual becomes functionally disabled. Here, $c > 1$ is defined as a constant reflecting higher payoffs in dependency. The life care product comprises the life annuity benefits B_0^h supplemented with $c \times B_0^h$ as an LTC cover specified at time 0 to provide a higher payoff in the event of a participant becoming functionally disabled. Since it is possible to recover from functional disability, if the individual becomes healthy after being functionally disabled, the annuity benefits are reduced from B_0^f to B_0^h. The predefined benefit amounts are constant over the insured lifetime.

However, in a pooled annuity design, the annuity benefits for both healthy and functionally disabled participants are not fixed in the future. Instead, they are updated based on the realized mortality and morbidity experience. We refer to a life care product in a pooled annuity design, a pooled health care annuity. The proposed product aims to provide, on average, the same payout as the standard life care annuity described above.

Assume at the start, there are $l_x^{h^*}$ healthy and $l_x^{f^*}$ functionally disabled pool participants aged x. This assumption is critical for our modelling framework since we focus on pooling mortality risk across heterogeneous individuals; thus, each pool must have at least one participant in the beginning. Similar assumptions were considered in Chen et al. (2022) for care-dependent tontines.

At time 0, two sets of annuity benefits can be expressed in a matrix form as follows:

$$\mathbf{B}_0 = \begin{bmatrix} B_0^h \\ \\ B_0^f \end{bmatrix} = \begin{bmatrix} B_0^h \\ \\ (1+c) \times B_0^h \end{bmatrix}, \tag{2.21}$$

where $\mathbf{B}_0$ is a vector of annuity benefits B_0^h and B_0^f, representing the payout to a healthy and functionally disabled pool participant, respectively at time 0.

For the three-state health transition model, the expected one year transition probabilities of a healthy and functionally disabled person at time 0 can be presented with the following matrix

$$\mathbf{P}_x = \begin{bmatrix} p_x^{hh} & p_x^{hf} \\ \\ p_x^{fh} & p_x^{ff} \end{bmatrix}. \tag{2.22}$$

Also, the matrix of the corresponding actuarial values for individuals in different state is

$$\mathbf{A}_x = \begin{bmatrix} a_x^{hh} & a_x^{hf} \\ \\ a_x^{fh} & a_x^{ff} \end{bmatrix}, \tag{2.23}$$

where the actuarial values $a_x^{hh} = \sum_0^{\omega-x} \upsilon^t {}_tp_x^{hh}$, $a_x^{hf} = \sum_0^{\omega-x} \upsilon^t {}_tp_x^{hf}$, $a_x^{ff} = \sum_0^{\omega-x} \upsilon^t {}_tp_x^{ff}$ and $a_x^{fh} = \sum_0^{\omega-x} \upsilon^t {}_tp_x^{hf}$ are defined based on the transition probabilities in healthy and functional disability states.

The fund values at time 0 can be decomposed into two groups as

$$\begin{aligned} F_0^h &= l_x^{h^*} \left(a_x^{hh} B_0^h + a_x^{hf} B_0^f \right), \\ F_0^f &= l_x^{f^*} \left(a_x^{fh} B_0^h + a_x^{ff} B_0^f \right), \end{aligned} \tag{2.24}$$

where F_0^h and F_0^f are the total starting fund values for healthy and functionally disabled pool participants. From Equation (2.24), a group of healthy participants contribute F_0^h initially to receive B_0^h as long as they remain healthy and B_0^f if they become functionally disabled at some point in the future, whereas the functionally disabled pool participants contribute F_0^f to receive B_0^f as long as they remain functionally disabled and B_0^h if they recover from functional disability at future times.

In the subsequent periods, the evolution of the fund value for individuals in a given health state depends on several factors, including the observed number of individuals transitioning in and out of the state, the fund balance at the start, and the investment earnings.

The classical method for solving for the fund value or reserves dynamics involves the solution of a discrete-time version of Thiele's Differential Equation as discussed in Hoem (1969). For example, in the three-state functional disability model with recovery, annuity payments for individuals in a given state are calculated based on the corresponding fund value at the start of the year, with transitions occurring at the end of the period. The fund balance for individuals in a specific state is calculated by adding the fund value after subtracting benefits paid to individuals in that state at the beginning of the period, then adding the fund value for individuals transitioning into the state over one period and subtracting the fund value required to determine benefits for individuals transitioning out of the state after investment earnings. In our setting, we model the fund dynamics using the matrix approach recently presented in Bladt et al. (2020). This approach determines the fund value in a given state by a cross product of the matrices of fund values, the number of participants and the actuarial values as shown in Equation (2.25) and it is consistent with Hoem (1969). We start by representing Equation (2.24) in a matrix form as

$$\begin{bmatrix} F_0^h \\ F_0^f \end{bmatrix} = \begin{bmatrix} l_x^{h^*} \\ l_x^{f^*} \end{bmatrix} \otimes \left(\begin{bmatrix} a_x^{hh} & a_x^{hf} \\ a_x^{fh} & a_x^{ff} \end{bmatrix} \begin{bmatrix} B_0^h \\ B_0^f \end{bmatrix} \right), \tag{2.25}$$

$$\mathbf{F}_0 = \mathbf{L}_x \otimes (\mathbf{A}_x \mathbf{B}_0), \tag{2.26}$$

where $\otimes$ denotes element-wise multiplication (also known as the Hadamard product) for multiplying each row of the column matrix $\mathbf{L}_x$ to each row of the matrix $\mathbf{A}_x\mathbf{B}_0$; $\mathbf{F}_0$ is a 2×1 vector of the pool fund values for healthy and functionally disabled participants, denoted by F_0^h and F_0^f respectively; $\mathbf{L}_x$ is a 2×1 vector of the actual number of healthy and disabled participants denoted by $l_x^{h^*}$ and $l_x^{f^*}$ respectively; and $\mathbf{A}_0$ is a 2×2 matrix of the actuarial values a_x^{hh}, a_x^{hf}, a_x^{fh}, a_x^{ff} in time 0.

The pool fund value at time 1 can be presented using the following equation

$$\mathbf{F}_1 = \mathbf{L}_{x+1} \otimes \left((\mathbf{P}_x^*)^{-1} \left(\frac{\mathbf{F}_0 - \mathbf{L}_x \otimes \mathbf{B}_0}{\mathbf{L}_x} \right) (1 + R_1) \right), \tag{2.27}$$

where

$$\mathbf{P}_x^* = \begin{bmatrix} p_x^{hh^*} & p_x^{hf^*} \\ p_x^{fh^*} & p_x^{ff^*} \end{bmatrix} \tag{2.28}$$

and

$$\mathbf{L}_{x+1} = \begin{bmatrix} l_{x+1}^{h^*} \\ l_{x+1}^{f^*} \end{bmatrix} \tag{2.29}$$

are the matrices of the realized one year transition probabilities and the number of participants, respectively at time 1. The realized transition probabilities are defined as $p_x^{hh^*} = \frac{l_{x+1}^{hh^*}}{l_x^{h^*}}$, $p_x^{hf^*} = \frac{l_{x+1}^{hf^*}}{l_x^{h^*}}$, $p_x^{fh^*} = \frac{l_{x+1}^{fh^*}}{l_x^{f^*}}$ and $p_x^{ff^*} = \frac{l_{x+1}^{ff^*}}{l_x^{f^*}}$. Here, $\frac{\mathbf{A}}{\mathbf{B}}$ is the element-wise division of matrices $\mathbf{A}$ and $\mathbf{B}$.

The actual number of healthy and functionally disabled participants at time 1 denoted by $l_{x+1}^{h^*}$ and $l_{x+1}^{f^*}$ respectively, is determined as follows

$$
\begin{aligned}
l_{x+1}^{h^*} &= l_x^{h^*} + l_{x+1}^{fh^*} - l_{x+1}^{hf^*} - l_{x+1}^{hd^*},\\
l_{x+1}^{f^*} &= l_x^{f^*} + l_{x+1}^{hf^*} - l_{x+1}^{fh^*} - l_{x+1}^{fd^*},
\end{aligned}
\tag{2.30}
$$

where $l_{x+1}^{hf^*}$ is the observed number of healthy participants who have become functionally disabled, $l_{x+1}^{fh^*}$ is the observed number of functionally disabled participants who have recovered, $l_{x+1}^{fd^*}$ is the observed number of functionally disabled participants who have died and $l_{x+1}^{hd^*}$ is the observed number of healthy participants who have died over a one year period.

We also derive the mortality and disability credits for healthy and functionally disabled participants in one-period setting. For the three-state functional disability model with recovery, that is when $p_x^{fh} > 0$, the individual's fund value at time 1 is:

$$
\begin{bmatrix} F_{k,1}^h \\ F_{k,1}^f \end{bmatrix} = \begin{bmatrix} p_x^{hh^*} & p_x^{hf^*} \\ p_x^{fh^*} & p_x^{ff^*} \end{bmatrix}^{-1} \begin{bmatrix} F_{k,0}^h(1+R_1) \\ F_{k,0}^f(1+R_1) \end{bmatrix},
\tag{2.31}
$$

$$
\begin{bmatrix} F_{k,1}^h \\ F_{k,1}^f \end{bmatrix} = \frac{1}{\left(p_x^{hh^*} p_x^{ff^*} - p_x^{fh^*} p_x^{hf^*}\right)} \begin{bmatrix} p_x^{ff^*} & -p_x^{hf^*} \\ -p_x^{fh^*} & p_x^{hh^*} \end{bmatrix} \begin{bmatrix} F_{k,0}^h(1+R_1) \\ F_{k,0}^f(1+R_1) \end{bmatrix},
\tag{2.32}
$$

The mortality and disability credits are given as:

$$
\begin{aligned}
F_{k,1}^h &= \underbrace{\frac{p_x^{ff^*}}{p_x^{hh^*} p_x^{ff^*} - p_x^{fh^*} p_x^{hf^*}}}_{\text{Mortality credits}} F_{k,0}^h(1+R_1) - \underbrace{\frac{p_x^{hf^*}}{p_x^{hh^*} p_x^{ff^*} - p_x^{fh^*} p_x^{hf^*}}}_{\text{Disability credits}} F_{k,0}^f(1+R_1),\\
F_{k,1}^f &= \underbrace{\frac{p_x^{hh^*}}{p_x^{hh^*} p_x^{ff^*} - p_x^{fh^*} p_x^{hf^*}}}_{\text{Mortality credits}} F_{k,0}^f(1+R_1) - \underbrace{\frac{p_x^{fh^*}}{p_x^{hh^*} p_x^{ff^*} - p_x^{fh^*} p_x^{hf^*}}}_{\text{Disability credits}} F_{k,0}^h(1+R_1).
\end{aligned}
\tag{2.33}
$$

Individuals have mortality and disability credits calculated based on transitions to and from the functional disability state as well as transitions to the death state.

To simplify the analysis and interpretation, we also consider the three-state functional disability model without recovery, that is when $p_x^{fh} = 0$. The individual's fund value in time 1 is:

$$
\begin{bmatrix} F_{k,1}^h \\ F_{k,1}^f \end{bmatrix} = \begin{bmatrix} p_x^{hh^*} & p_x^{hf^*} \\ 0 & p_x^{ff^*} \end{bmatrix}^{-1} \begin{bmatrix} F_{k,0}^h(1+R_1) \\ F_{k,0}^f(1+R_1) \end{bmatrix},
\tag{2.34}
$$

$$
\begin{bmatrix} F_{k,1}^h \\ F_{k,1}^f \end{bmatrix} = \frac{1}{p_x^{hh^*} p_x^{ff^*}} \begin{bmatrix} p_x^{ff^*} & -p_x^{hf^*} \\ 0 & p_x^{hh^*} \end{bmatrix} \begin{bmatrix} F_{k,0}^h(1+R_1) \\ F_{k,0}^f(1+R_1) \end{bmatrix}.
\tag{2.35}
$$

The two groups are distinguished as follows:

$$
F_{k,1}^h = \frac{p_x^{ff^*}}{p_x^{hh^*} p_x^{ff^*}} F_{k,0}^h(1+R_1) - \frac{p_x^{hf^*}}{p_x^{hh^*} p_x^{ff^*}} F_{k,0}^f(1+R_1),
\tag{2.36}
$$

$$F^f_{k,1} = \frac{p_x^{hh^*}}{p_x^{hh^*} p_x^{ff^*}} F^f_{k,0}(1+R_1). \tag{2.37}$$

Equations (2.36) and (2.37) can be written as

$$\begin{aligned} F^h_{k,1} &= \underbrace{\frac{1}{p_x^{hh^*}}}_{\text{Mortality credits}} F^h_{k,0}(1+R_1) - \underbrace{\frac{p_x^{hf^*}}{p_x^{hh^*} p_x^{ff^*}}}_{\text{Disability credits}} F^f_{k,0}(1+R_1), \\ F^f_{k,1} &= \underbrace{\frac{1}{p_x^{ff^*}}}_{\text{Mortality credits}} F^f_{k,0}(1+R_1). \end{aligned} \tag{2.38}$$

This means that for the healthy group, the fund value is adjusted by considering transitions to functional disability and death states, crediting the fund value for those who transit to the functional disability state and adding any mortality credits within the group. However, for the functionally disabled group, there is no need to adjust fund value for transitions to the healthy group since there is no recovery from functional disability. Therefore, the fund value for the functionally disabled is adjusted by only considering the mortality credits within the group. But, we know that in general, $p_x^{hh^*} > p_x^{ff^*}$ hence we expect to have higher mortality credits in the functional disabled group than in the healthy group. This result is similar to the discussions in Hieber and Lucas (2022) and Chen et al. (2022) in a continuous time setting.

From our model setting in Equation (2.25), the pooled health care annuity payouts for individuals in all states at time 1 can be determined as

$$\mathbf{F}_1 = \mathbf{L}_{x+1} \otimes (\mathbf{A}_{x+1}\mathbf{B}_1), \tag{2.39}$$

where $\mathbf{A}_{x+1} = \begin{bmatrix} a^{hh}_{x+1} & a^{hf}_{x+1} \\ a^{fh}_{x+1} & a^{ff}_{x+1} \end{bmatrix}$ is a matrix of the actuarial values a^{hh}_{x+1}, a^{hf}_{x+1}, a^{fh}_{x+1}, a^{ff}_{x+1} in time 1.

Recursively, the fund value can be determined using the following formula

$$\mathbf{F}_{t+1} = \mathbf{L}_{x+t+1} \otimes \left((\mathbf{P}^*_{x+t})^{-1} \left(\frac{\mathbf{F}_t - \mathbf{L}_{x+t} \otimes \mathbf{B}_t}{\mathbf{L}_{x+t}} \right) (1+R_t) \right), \tag{2.40}$$

where $\mathbf{F}_t = \begin{bmatrix} F^h_t \\ F^f_t \end{bmatrix}$, $\mathbf{L}_{x+t} = \begin{bmatrix} l^{h^*}_{x+t} \\ l^{f^*}_{x+t} \end{bmatrix}$, $\mathbf{B}_t = \begin{bmatrix} B^h_t \\ B^f_t \end{bmatrix}$, $\mathbf{P}^*_{x+t} = \begin{bmatrix} p^{hh^*}_{x+t} & p^{hf^*}_{x+t} \\ p^{fh^*}_{x+t} & p^{ff^*}_{x+t} \end{bmatrix}$, and R_t is the actual interest earning at time t.

The pooled health care annuity benefits $\mathbf{B}_{t+1}$ are derived by solving the system of simultaneous equations from the following relationship,

$$\mathbf{F}_{t+1} = \mathbf{L}_{x+t+1} \otimes (\mathbf{A}_{x+t+1}\mathbf{B}_{t+1}), \tag{2.41}$$

given that $\mathbf{L}_{x+t+1} = \begin{bmatrix} l^{h^*}_{x+t+1} \\ l^{f^*}_{x+t+1} \end{bmatrix}$ and $\mathbf{A}_{x+t+1} = \begin{bmatrix} a^{hh}_{x+t+1} & a^{hf}_{x+t+1} \\ a^{fh}_{x+t+1} & a^{ff}_{x+t+1} \end{bmatrix}$.

The realized one year transition probabilities are defined as $p^{hh^*}_{x+t} = \frac{l^{hh^*}_{x+t+1}}{l^{h^*}_{x+t}}$, $p^{hf^*}_{x+t} = \frac{l^{hf^*}_{x+t+1}}{l^{h^*}_{x+t}}$, $p^{fh^*}_{x+t} = \frac{l^{fh^*}_{x+t+1}}{l^{f^*}_{x+t}}$ and $p^{ff^*}_{x+t} = \frac{l^{ff^*}_{x+t+1}}{l^{f^*}_{x+t}}$ where

$$\begin{aligned} l^{h^*}_{x+t+1} &= l^{h^*}_{x+t} + l^{fh^*}_{x+t+1} - l^{hf^*}_{x+t+1} - l^{hd^*}_{x+t+1}, \\ l^{f^*}_{x+t+1} &= l^{f^*}_{x+t} + l^{hf^*}_{x+t+1} - l^{fh^*}_{x+t+1} - l^{fd^*}_{x+t+1}. \end{aligned} \tag{2.42}$$

The matrix $\mathbf{P}^*_{x+t}$ is used to distribute morbidity and mortality credits. Morbidity credits are required to adjust the benefits of individuals who have transitioned to the functional disability state. These can be positive or negative depending on the actual number of participants transitioning to that state at a given point in time.

2.4 A five-state model pooling structure

In the three-state functional disability model described in Subsection 2.3, individuals are not distinguished based on their health status, particularly chronic illness. For example, individuals who are not functionally disabled are classified as healthy and fall into the same risk category regardless of whether they are healthy or sick. In practice, a non-disabled person can be unhealthy. Sherris and Wei (2021) considered a five-state model where each individual is further classified along with another possible dimension: health (other than disability) status. Their classification results in five different states: Good health and not functionally disabled (H); Ill health and not functionally disabled (M); Good health and functionally disabled (F); Ill health and functionally disabled (MF); and Dead (D). Recovery from a disability is permitted, whereas recovery from ill health is not included since the medical history of chronic illnesses is incorporated.

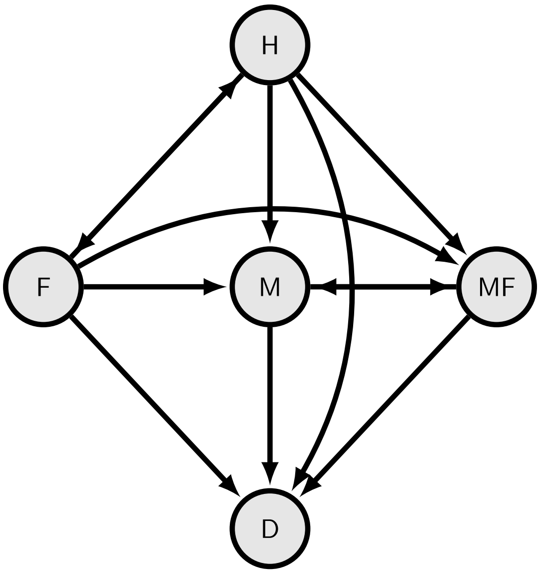


Figure 3: A five-state functional disability and health status model allowing for recovery in functional disability.

Figure 3 illustrates the functional disability and health status model proposed in Sherris and Wei (2021). There are twelve types of transitions: $H \to M$, $H \to F$, $H \to MF$, $H \to D$, $M \to MF$, $M \to D$, $F \to H$, $F \to M$, $F \to MF$, $F \to D$, $MF \to M$ and $MF \to D$. We define the following additional one year transition probabilities for an individual aged x:

- p_x^{hm}: the probability that a healthy person aged x becomes sick at age $x+1$;
- $p_x^{h,mf}$: the probability that a healthy person aged x becomes sick and functionally disabled at age $x+1$;
- p_x^{fm}: the probability that a functionally disabled person aged x at time t becomes sick but recovers from disability at age $x+1$;
- $p_x^{f,mf}$: the probability that a functionally disabled person aged x becomes sick and remains functionally disabled at age $x+1$;

- p_x^{mm}: the probability that a sick person aged $x+t$ remains sick at age $x+1$;
- $p_x^{m,mf}$: the probability that a sick person aged $x+t$ remains sick and becomes functionally disabled at age $x+1$;
- p_x^{md}: the probability that a sick person aged $x+t$ dies before age $x+1$;
- $p_x^{mf,mf}$: the probability that a sick and functionally disabled person aged x remains sick and functionally disabled at age $x+1$;
- $p_x^{mf,m}$: the probability that a sick and functionally disabled person aged x remains sick but recovers from disability at age $x+1$;
- $p_x^{mf,d}$: the probability that a sick and functionally disabled person aged $x+t$ dies before reaching age $x+1$.

We apply this model to further distinguish pool participants according to chronic illness status and demonstrate how different individuals can share mortality, functional disability, and health risk in a pool. Next, we extend the pooled health care annuity design in the three-state framework (described in Subsection 2.3) to derive the annuity benefits in the five-state context.

For the five-state functional disability and health status model, the annuity benefits are paid in four states namely: H, M, F and MF. We define four types of annuity benefits at time 0; B_0^h, B_0^m, B_0^f and B_0^{mf}, representing the benefits amount for a participants in good health without functional disability, ill health without functional disability, good health with functional disability as well as ill health with functional disability, respectively. In a similar manner, the benefits in worse health conditions are defined in terms of an uplift with respect to the annuity benefits in healthy condition using constants c_1, c_2 and c_3. According to the degree of dependency, benefits in the ill health state are defined as $B_0^m = (1+c_1) \times B_0^h$, while in good health with functional disability $B_0^f = (1+c_2) \times B_0^h$ and in ill health with functional disability $B_0^{mf} = (1+c_3) \times B_0^h$, here $c_1 < c_2 < c_3$. The individual's benefits at time 0 are presented in a vector form as follows

$$\mathbf{B}_0 = \begin{bmatrix} B_0^h \\ \\ B_0^m \\ \\ B_0^f \\ \\ B_0^{mf} \end{bmatrix} = \begin{bmatrix} B_0^h \\ \\ (1+c_1) \times B_0^h \\ \\ (1+c_2) \times B_0^h \\ \\ (1+c_3) \times B_0^h \end{bmatrix}, \tag{2.43}$$

where $\mathbf{B}_0$ is a vector of the annuity benefits in different health states.

Based on the four health states in the functional disability and health status model, the transition probability matrix for a surviving person aged x is defined as

$$\mathbf{P}_x = \begin{bmatrix} p_x^{hh} & p_x^{hm} & p_x^{hf} & p_x^{h,mf} \\ \\ 0 & p_x^{mm} & 0 & p_x^{m,mf} \\ \\ p_x^{fh} & p_x^{fm} & p_x^{ff} & p_x^{f,mf} \\ \\ 0 & p_x^{mf,m} & 0 & p_x^{mf,mf} \end{bmatrix}, \tag{2.44}$$

and the corresponding matrix of the actuarial present values defined based on the transition probabilities is

$$\mathbf{A}_x = \begin{bmatrix} a_x^{hh} & a_x^{hm} & a_x^{hf} & a_x^{h,mf} \\ 0 & a_x^{mm} & 0 & \ddot{a}_x^{m,mf} \\ a_x^{fh} & a_{x+t}^{fm} & a_x^{ff} & a_x^{f,mf} \\ 0 & a_x^{mf,m} & 0 & a_x^{mf,mf} \end{bmatrix}, \tag{2.45}$$

where the actuarial values $a_x^{hh} = \sum_0^{\omega-x} \upsilon^t {}_tp_x^{hh}$, $a_x^{hm} = \sum_0^{\omega-x} \upsilon^t {}_tp_x^{hm}$, $a_x^{hf} = \sum_0^{\omega-x} \upsilon^t {}_tp_x^{hf}$, $a_x^{h,mf} = \sum_0^{\omega-x} \upsilon^t {}_tp_x^{h,mf}$, $a_x^{mm} = \sum_0^{\omega-x} \upsilon^t {}_tp_x^{mm}$, $a_x^{m,mf} = \sum_0^{\omega-x} \upsilon^t {}_tp_x^{m,mf}$, $a_x^{fh} = \sum_0^{\omega-x} \upsilon^t {}_tp_x^{fh}$, $a_x^{fm} = \sum_0^{\omega-x} \upsilon^t {}_tp_x^{fm}$, $a_x^{ff} = \sum_0^{\omega-x} \upsilon^t {}_tp_x^{ff}$, $a_x^{mf,m} = \sum_0^{\omega-x} \upsilon^t {}_tp_x^{mf,m}$, as well as $a_x^{mf,mf} = \sum_0^{\omega-x} \upsilon^t {}_tp_x^{mf,mf}$ are defined based on the transition probabilities in the states of health, illness, and functional disability.

The pool fund value at initial time is

$$\begin{bmatrix} F_0^h \\ F_0^m \\ F_0^f \\ F_0^{mf} \end{bmatrix} = \begin{bmatrix} l_x^{h^*} \\ l_x^{m^*} \\ l_x^{f^*} \\ l_x^{mf^*} \end{bmatrix} \otimes \left(\begin{bmatrix} a_x^{hh} & a_x^{hm} & a_x^{hf} & a_x^{h,mf} \\ 0 & a_x^{mm} & 0 & \ddot{a}_x^{m,mf} \\ a_x^{fh} & a_{x+t}^{fm} & a_x^{ff} & a_x^{f,mf} \\ 0 & a_x^{mf,m} & 0 & a_x^{mf,mf} \end{bmatrix} \begin{bmatrix} B_0^h \\ B_0^m \\ B_0^f \\ B_0^{mf} \end{bmatrix} \right), \tag{2.46}$$

$$\mathbf{F}_0 = \mathbf{L}_x \otimes (\mathbf{A}_x \mathbf{B}_0), \tag{2.47}$$

where $\mathbf{F}_0$ is a 4×1 matrix of the pool fund value for those in good and ill health with and without functional disability and $\mathbf{L}_x$ is a 4×1 matrix of the initial number of participants aged x at time 0. $\mathbf{A}_x$ is a 4×4 matrix of actuarial values that are defined based on the transition probabilities in good and ill health.

As with the three-state model, the dynamics of the pool fund value in time t are presented as

$$\mathbf{F}_{t+1} = \mathbf{L}_{x+t+1} \otimes \left((\mathbf{P}_{x+t}^*)^{-1} \left(\frac{\mathbf{F}_t - \mathbf{L}_{x+t} \otimes \mathbf{B}_t}{\mathbf{L}_{x+t}} \right) (1 + R_t) \right), \tag{2.48}$$

where $\mathbf{F}_{t+1}$ is a matrix of the pool fund values for participants in good and ill health with and without functional disability, $\mathbf{L}_{x+t+1}$ is a matrix of the actual number of participants in different states at time $t+1$, $\mathbf{P}_{x+t}^*$ is a matrix of the realized transition probabilities, $\mathbf{L}_{x+t}$ is a matrix of the actual number of participants in different states and $\mathbf{B}_t$ is matrix of the annuity benefits at time t.

The matrices are defined as follows $\mathbf{F}_{t+1} = \begin{bmatrix} F_{t+1}^h \\ F_{t+1}^m \\ F_{t+1}^f \\ F_{t+1}^{mf} \end{bmatrix}$, $\mathbf{L}_{x+t+1} = \begin{bmatrix} l_{x+t+1}^{h^*} \\ l_{x+t+1}^{m^*} \\ l_{x+t+1}^{f^*} \\ l_{x+t+1}^{mf^*} \end{bmatrix}$, and

$$\mathbf{P}_{x+t} = \begin{bmatrix} p_{x+t}^{hh} & p_{x+t}^{hm} & p_{x+t}^{hf} & p_{x+t}^{h,mf} \\ 0 & p_{x+t}^{mm} & 0 & p_{x+t}^{m,mf} \\ p_{x+t}^{fh} & p_{x+t}^{fm} & p_{x+t}^{ff} & p_{x+t}^{f,mf} \\ 0 & p_{x+t}^{mf,m} & 0 & p_{x+t}^{mf,mf} \end{bmatrix}.$$

The matrix of the pooled health care annuity benefits $\mathbf{B}_{t+1}$ can be derived by solving the system of simultaneously equations from the following

$$\mathbf{F}_{t+1} = \mathbf{L}_{x+t+1} \otimes (\mathbf{A}_{x+t+1} \mathbf{B}_{t+1}), \tag{2.49}$$

where

$$\mathbf{A}_{x+t+1} = \begin{bmatrix} a_{x+t+1}^{hh} & a_{x+t+1}^{hm} & a_{x+t+1}^{hf} & a_{x+t+1}^{h,mf} \\ 0 & a_{x+t+1}^{mm} & 0 & \ddot{a}_{x+t+1}^{m,mf} \\ a_{x+t+1}^{fh} & a_{x+t}^{fm} & a_{x+t+1}^{ff} & a_{x+t+1}^{f,mf} \\ 0 & a_{x+t+1}^{mf,m} & 0 & a_{x+t+1}^{mf,mf} \end{bmatrix}. \tag{2.50}$$

2.5 Multi-state models estimation

To demonstrate the concept of mortality risk pooling using multi-state models, we must compute the estimated transition probabilities. Using maximum likelihood estimation techniques, we first estimate the parameters of the multi-state models. The transition probabilities are then calculated by simulating health transition rates using the fitted models. Finally, the actuarial values are calculated to determine the pooled health care annuity benefits. According to Pitacco (2014), the process from raw data to actuarial present values involves three steps:

1. Graduation of observed raw data to obtain transition intensities (rates).
2. Obtaining transition probabilities from transition intensities.
3. Application of transition probabilities in valuation formulas and computation of actuarial values to obtain premiums and reserves.

We estimate the three-state functional disability model with recovery and the five-state functional disability and health status model. These models have been estimated in Fu et al. (2021) and Sherris and Wei (2021), respectively, based on individual-level data from the Health and Retirement Study covering the US elderly population. We estimate the models based on female and male observations to utilize the data effectively.[2] However, in numerical examples, we only use estimated model results for males to describe the proposed mortality risk pooling framework. We apply the estimation procedures proposed in Fu et al. (2021) which improve the numerical stability in the estimated parameters. Sherris and Wei (2021) and Fu et al. (2021) both adopt a proportional hazard specification, similar to that used in Koopman et al. (2008) for credit-rating transitions.

For an individual k at time t, the transition intensity for transition type s is considered to be of the form

$$\lambda_{k,s}(t) = \exp\left(\beta_s + \gamma_s^{'} w_k(t) + \alpha_s \psi(t)\right), \tag{2.51}$$

where β_s is the baseline log-intensity for transition type s, independent of time and common across all individuals. The vector $w_k(t)$ contains the observed predictors for each individual k, and is restricted to age, gender and time. $\psi(t)$ is a stochastic latent process that drives systematic uncertainties, also known as frailty. The parameter vector γ_s and scalar α_s measure the sensitivities of logarithm of $\lambda_{k,s}(t)$, with respect to $w_k(t)$ and $\psi(t)$. For tractability, the transition rates are assumed to be piece-wise constant at integer ages.

The following notations are used to present the exact functional form of the piece wise constant transition rates:

- s, s^{th} transition type, $s = 1, 2, ..., S$;
- k, k^{th} individual, $k = 1, 2, ..., K$;
- G_j, j^{th} individual's gender, $G_j = 1$ if the j^{th} individual is female and 0 otherwise;
- v, v^{th} interview, $v = 1, 2, ..., V$;
- t, time measured in years;
- $x_k(t)$, k^{th} individual's age at time t;
- $t_{k,v}$, the time of the v^{th} interview for the k^{th} individual;
- $\hat{t}_{k,v}$, the time of transition between the v^{th} and $v+1^{st}$ interviews for the k^{th} individual, should it occur; $\hat{t}_{k,v}$ is the exact death time if the k^{th} individual died during this period; otherwise, $(t_{k,v} + t_{k,v+1})/2$, the mid-point of the time between the v^{th} and $(v+1)^{st}$ interviews for the k^{th} individual.

[2]We have used the guidelines provided in Fu et al. (2021) for estimating functional disability models and codes available at `https://sites.google.com/view/mxu/code`.

For the three-state functional disability model, the state space of a transition type is $s = \{1, \ldots, 4\}$. We set $s = 1$ for $H \to F$, $s = 2$ for $F \to H$, $s = 3$ for $H \to D$ and $s = 4$ for $F \to D$. For the five-state functional disability and health status model, the state space is $s = \{1, \ldots, 12\}$ where we denote $s = 1$ for $H \to M$, $s = 2$ for $H \to F$, $s = 3$ for $H \to MF$, $s = 4$ for $H \to D$, $s = 5$ for $M \to MF$, $s = 6$ for $M \to D$, $s = 7$ for $F \to H$, $s = 8$ for $F \to M$, $s = 9$ for $F \to MF$, $s = 10$ for $F \to D$, $s = 11$ for $MF \to M$ and $s = 12$ for $MF \to D$.

Three models are considered in estimating parameters of the multi-state models: a static or no-frailty model, a trend or no-frailty model with a linear time trend, and a frailty model.

1. In the static model, the transition rate $\lambda_{k,s}(t)$ is assumed to be dependent on age and gender only:

$$\ln \lambda_{k,s}(t) = \beta_s + \gamma_s^{\text{age}} x_k(t) + \gamma_s^{\text{female}} G_k, \tag{2.52}$$

   where β_s is the reference level of $\lambda_{k,s}(t)$ that varies by transition type, $x_k(t)$ is the age of the k^{th} individual at time t, and G_k is an indicator variable that indicates whether the k^{th} individual is female or male. The sensitivity of ln $\lambda_{k,s}(t)$ with respect to age and gender is measured by γ_s^{age} and γ_s^{female}, respectively.

2. To model the systematic time trend in $\lambda_{k,s}(t)$, the linear time index is included in the trend model:

$$\ln \lambda_{k,s}(t) = \beta_s + \gamma_s^{\text{age}} x_k(t) + \gamma_s^{\text{female}} G_k + \gamma_s^{\text{time}} t, \quad t_{k,v} \le t < t_{k,v+1}, \tag{2.53}$$

   where γ_s^{time} measures the the sensitivity of ln $\lambda_{k,s}(t)$ with respect to the time trend (wave index) t.

3. Then, for the frailty model with time trend, the latent factor $\psi(t)$ is added to ln $\lambda_{k,s}(t)$, to account for systematic uncertainty:

$$\ln \lambda_{k,s}(t) = \beta_s + \gamma_s^{\text{age}} x_k(t) + \gamma_s^{\text{female}} G_k + \gamma_s^{\text{time}} t + \alpha_s \psi(t), \quad t_{k,v} \le t < t_{k,v+1}, \tag{2.54}$$

   where α_s measures the the sensitivity of ln $\lambda_{k,s}(t)$ with respect to the latent factor. The latent factor $\psi(t)$ is modeled as a simple random walk with drift term as follows:

$$\psi(t) = \rho^{t_v - t_{v-1}} \psi_{v-1} + \epsilon_v, \ \epsilon_v \sim NIID(0, 1), \ \psi_0 = 0, \tag{2.55}$$

   where $\rho^{t_v - t_{v-1}} = 1$ for a random walk process and t_v measured in years denotes the time of the v^{th} interview. We denote $\psi_v = \psi(t_v)$ as the value of $\psi(t)$ over the interval $t \in (t_{v-1}, t_v]$.

Maximum likelihood estimation techniques are used to determine the parameters of the static and trend (no-frailty) models, and the frailty process is recovered using a Kalman filter and smoother. After estimating the parameters, we use the Kalman filtering and smoothing technique to recover the frailty process. Sherris and Wei (2021) gives more details about the Kalman filtering and smoothing technique.

2.5.1 Data description

The Health and Retirement Study (2021) is a longitudinal household survey of people aged 50 in the US that began in 1992. The survey includes questions on the respondent's health history, physical and cognitive status. In particular, the RAND HRS Longitudinal File 2018 (V1) (2021) has fourteen interviews for sixteen survey years; 1992, 1993, 1994, 1995, and biennially 1996-2018. We use data from Wave 1998 onward as the survey questions were inconsistent prior to Wave 1998; similar assumptions were made in Fu et al. (2021) and Sherris and Wei (2021). The previous research (Fu et al. (2021) and Sherris and Wei (2021)) used data from 1992 to 2014, we re-estimate the models using data from Waves 1998 to 2018.

To illustrate the mortality sharing arrangement based on the two multi-state models, we first need to create transition datasets that comprise transitions between various health states. We create two transition datasets. The first dataset contains three-state transitions for the functional disability model, and the second dataset includes five-state transitions for the functional disability and health status model. An individual is classified as functionally disabled if they have two or more difficulties in any of six activities of daily living (ADLs), namely dressing, walking, bathing, eating, transferring and toileting. Also, each individual is categorized as in good health or ill health depending on whether they have ever had one of the following diseases: heart problems, diabetes, lung disease, and stroke. In creating these datasets, we assume that transitions (other than death) occur at the midpoint of the time between the two interview dates since the exact time of transition was not stated during the survey. However, the exact time of death is available; we use the date of death for the transitions including deaths. We also assume that the transitions between states are interval-censored and updated on each birthday.

Before any data cleaning, there were 42,233 respondents from the fourteen interviews from 1992 to 2018. After removing individuals with missing interview dates, missing or invalid death dates, or missing information on ADLs, we are left with 36,129 people for the interview years 1998 to 2017. We excluded transitions that occurred in 2018 and individuals who were not interviewed between two consecutive waves. Our selected sample has more participants and more interview waves than the samples in Fu et al. (2021) and Sherris and Wei (2021).

3 Main Results

We estimate the parameters of the multi-state models from the transition datasets and simulate health transition rates using the fitted models. From the simulated transition rates, we estimate transition probabilities, actuarial values, pooled fund values, and annuity payouts for individuals in different health states. For simplicity, we only use estimated results for males.

Figure 4 shows the estimated transition probabilities of a 65-years old male associated with the three-state functional disability static model. The occupancy probability in the healthy state ${}_tp_{65}^{hh}$ starts at one, decreases through time as the person approaches death at older ages and eventually reaches zero. The occupancy probability in a functional disability state ${}_tp_{65}^{ff}$ is convex. It declines rapidly from one and approaches zero gradually when a disabled person is about to die. In contrast, the probability of a 65-year-old male transitioning to a functional disability status ${}_tp_{65}^{hf}$ increases with time at first and then decreases over a long period.

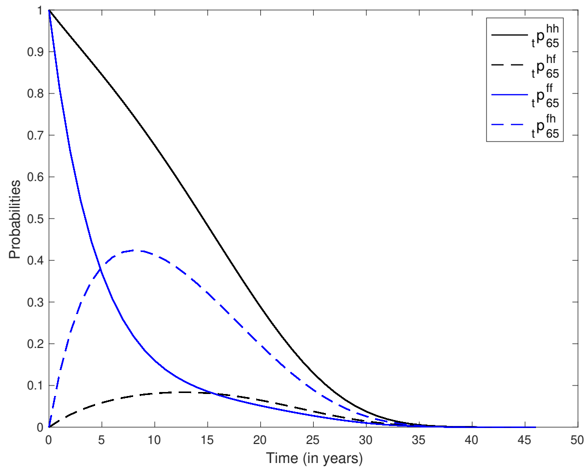
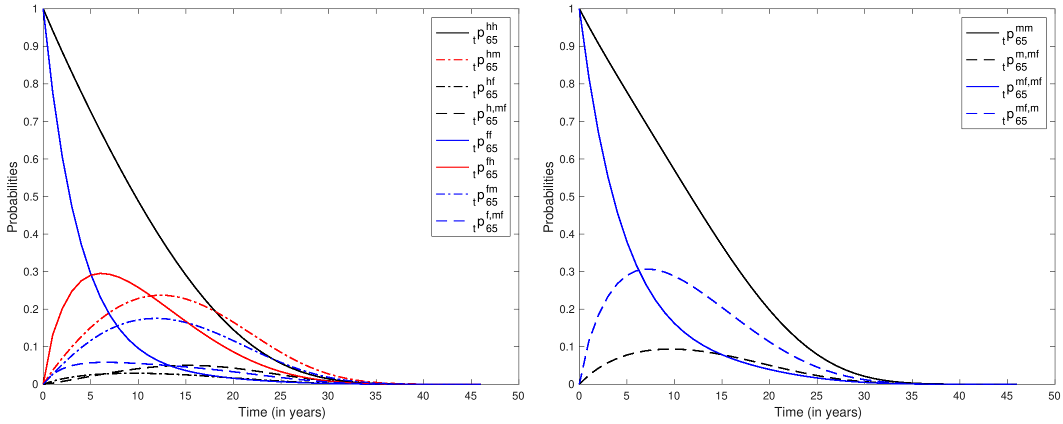


Figure 4: The estimated multi-year transition probabilities of a 65-year-old male from the static model based on the three-state functional disability model.

Interestingly, the recovery probability of a 65-years old male ${}_tp_{65}^{fh}$ is always higher than the transition



(a) Transition probabilities for a 65-year-old male in good health.

(b) Transition probabilities for a 65-year-old male in ill health.

Figure 5: The estimated multi-year transition probabilities of a 65-year-old male in good ans ill health from the static model based on the five-state functional disability and health status model.

probability $_tp_{65}^{hf}$, implying that it is more likely for a 65-year-old disabled male to recover from functional disability than a 65-year-old healthy male to become functionally disabled. This also contributes to the convex shape of $_tp_{65}^{ff}$, whereas in the early years, there is a higher possibility of exiting the functional disability state through recovery than death. In addition, $_tp_{65}^{fh}$ initially (from time 0 to approximately time 10) increases at a faster rate, then starts to decline rapidly. This implies that a disabled male aged 65 is more likely to recover from disability at a younger age than at older ages, necessitating the need for LTC insurance past age 75.

Figure 5 shows the transition probabilities of a 65-year-old male according to the five-state functional disability and health status static model. The left panel shows the transition probabilities for a 65-years old male in good health, while the right panel panel is for a 65-years old male in ill health. There are fewer transitions for an individual in poor health.

Similar to the three-state model, the occupancy probability in a health state $_tp_{65}^{hh}$ starts at one, decreases through time and eventually reaches zero. The occupancy probability in a functional disability state $_tp_{65}^{ff}$ forms a convex curve, declines rapidly from one and approaches zero gradually when a disabled person is about to die.

For an individual in ill health, $_tp_{65}^{mm}$ starts at one decrease through time and eventually reaches zero. The occupancy probability in functional disability state $_tp_{65}^{mf,mf}$ declines rapidly from one and approaches zero gradually. The recovery probability of a 65-years old male in good health $_tp_{65}^{fh}$ is greater than the transition probability to ill and functionally disability states ($_tp_{65}^{fm}$ and $_tp_{65}^{f,mf}$). For individuals in ill health, the recovery probability from disability $_tp_{65}^{mf,m}$ is also higher than the transition probability to functional disability $_tp_{65}^{m,mf}$. These findings demonstrate that the recovery probability from functional disability is not trivial regardless of a person's chronic illness status.

3.1 Numerical examples

This section presents the results obtained from numerical examples. We consider two numerical examples. The first represents a mortality risk pooling arrangement based on the three-state functional disability model, in which mortality risk is shared across heterogeneous individuals classified according to their functional disability states. The second example is based on the five-state functional disability and health status model, which further distinguishes participants based on their chronic illness status. We show the mean and the 95% confidence intervals of the number of participants in different states, the pool fund values, the individual's fund values and the corresponding annuity benefits depending on the individual's health state.

3.1.1 Numerical example 1

We examine the mortality risk pooling arrangement using the three-state functional disability model displayed in Figure 2. The base case initial pool size is set to 1,000 males aged 65 at time 0 (the calendar year 2018). We divide the initial pool into 920 healthy males and 80 disabled based on the percentages of the HRS interview respondents in 2017. The maximum age to receive the annuity benefits is 110. We set $c = 2$ in Equation (2.21) to provide three times healthy annuity benefits to the functionally disabled at time 0. This assumption has been considered in Sherris and Wei (2021) for the LTC policy. Also, in practice, the LTC benefits are usually three times the standard life annuity payouts. The annuity payout to a healthy participant is set to $B_0^h = \$12,000$ per year, and for the functionally disabled participant, is $B_0^f = \$36,000$ per year. The interest rate assumed in pricing is $r = 3\%$ per annum. The realized interest rate R_t for the evolution of the fund value at time t is the same as the expected rate used in pricing r, so we focus on mortality and disability risks.

We make several assumptions in determining the annuity payments. For the static model results, we use the estimated transition probabilities from the static model for pricing and evolution of the fund value. For the trend (frailty) model, we present two scenarios. In the first scenario, the annuity values of the trend (frailty) models are computed using the expected transition probabilities matrix of the static model, while the evolution of the fund value is determined using the trend (frailty) model. In this case, we use the static model as the expected assumptions in pricing and the trend (frailty) model as the realized assumptions. We illustrate what happens if we ignore time trends and systematic uncertainty in pricing or designing the pooled health care annuity product. In the second scenario, the annuity values are computed using the trend (frailty) model's estimated transition probabilities to include expected mortality and morbidity improvements in pricing, while the evolution of the fund value is also determined based on the trend (frailty) model.

Figure 6 shows the number of participants in healthy and disabled states, pool fund values, individual fund values, and pooled annuity payouts for the two groups using the static model. The top left panel of Figure 6 shows the realized number of healthy and disabled survivors using the static model, with an initial pool size of 920 healthy and 80 disabled participants. On average, both numbers decrease with time as individuals age. As previously stated, each pool must have at least one participant to keep operating the pooled health care plan. From age 96, the pool size falls below this threshold, and the pool fund value is depleted thus the plan stops since there are few participants to keep operating the pool. Participants therefore receive the annuity benefits for 31 years during the retirement.

On average, the pool fund value for the healthy group is higher than that of the disabled as there are more healthy participants than the disabled. In contrast, the corresponding individual's value for the healthy participant is lower than the disabled, as shown in the bottom left panel of Figure 6. A disabled participant has a higher fund value than a healthy participant to fund the increased payments in dependency.

During the retirement phase, the functionally disabled group also receives increased benefits due to the group's smaller pool size over long periods and higher mortality credits. The functionally disabled participants have higher mortality credits due to higher death probabilities than the healthy participants. On the other hand, there is a declining trend in the average benefit payments for the healthy group, which also affects the 95% percentile outcomes as the healthy participants receive lower mortality credits and there is large group size.

To better understand the difference between benefits for the healthy and functionally disabled participants, Figure 7 shows the estimated mortality and disability credits for the two groups. We used the results from the static model. We note that mortality credits increase over time for both groups due to the increased

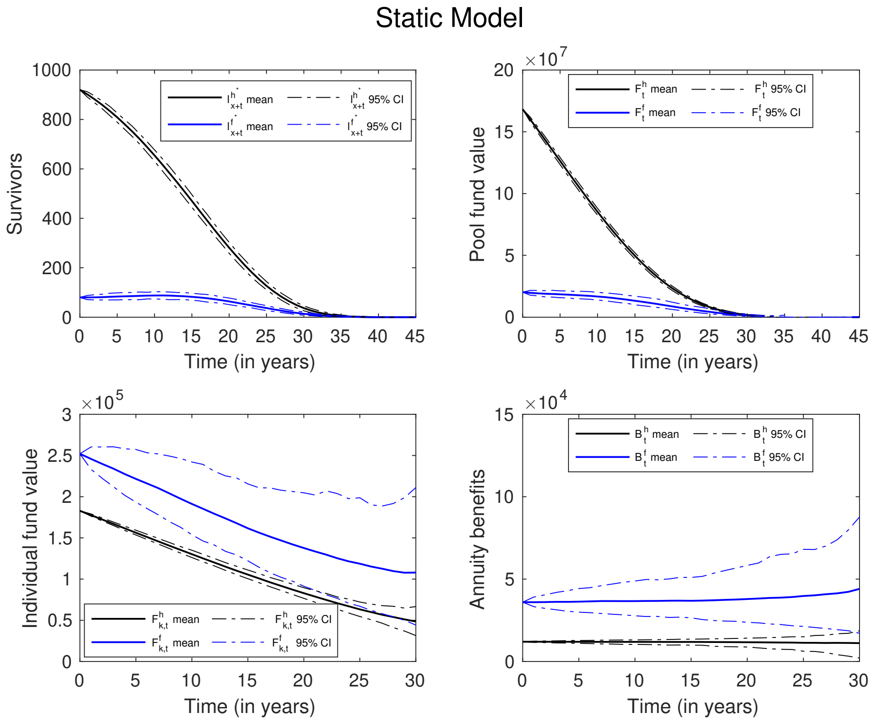


Figure 6: The static model's number of survivors in healthy and functional disability states, pool fund values, individual fund values and annuity benefits for healthy and disabled participants. The annuity values and the evolution in the fund values are computed using the static model which does not include trends and systematic uncertainties.

number of deaths as pool participants age. In terms of magnitude, the disabled participants receive higher mortality credits than the healthy participants since they have higher death probabilities due to disability.

On the other hand, we can see that they receive lower disability credits in most cases compared to the healthy group, especially in the early years. The disabled fund value is highly credited with providing benefits for those transitioning to the healthy group. In particular, at younger ages, many individuals recover from disability which reduces surplus credits.

The surplus or deficit in mortality and disability credits depends on the actual number of transitions. For example, if many individuals become functionally disabled, there is less surplus as it is more expensive to fund the disability benefits. In addition, the volatility in mortality and disability credits within the group depends on the group size: the smaller the group, the higher volatility.

From the trend model, Figure 8 shows the results when the time trend is not included in pricing. With no allowance for future expected mortality and morbidity improvements, benefits drop for the older survivors in the pool, as shown on the right bottom panel of Figure 8. The individual's fund value also decreases significantly with time for both groups since the static model used in pricing tends to overestimate mortality and disability rates, ignoring any possibilities for mortality improvement and morbidity compression.

From the frailty model, Figure 9 shows the results when systematic trends and uncertainty are not included in pricing. The benefits drop even more significantly for the older survivors in the pool than when only the time trend is ignored. Disregarding both systematic trends and uncertainty has a more significant impact on healthy participants than on disabled participants.

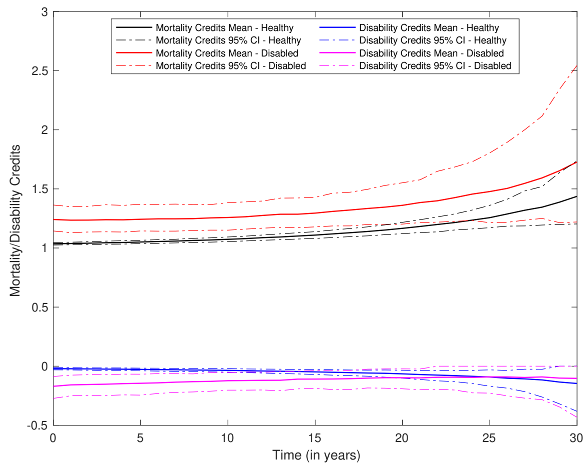


Figure 7: Mortality and disability credits over time for healthy and functionally disabled pool participants based on three-state pooling structure using the functional disability static model.

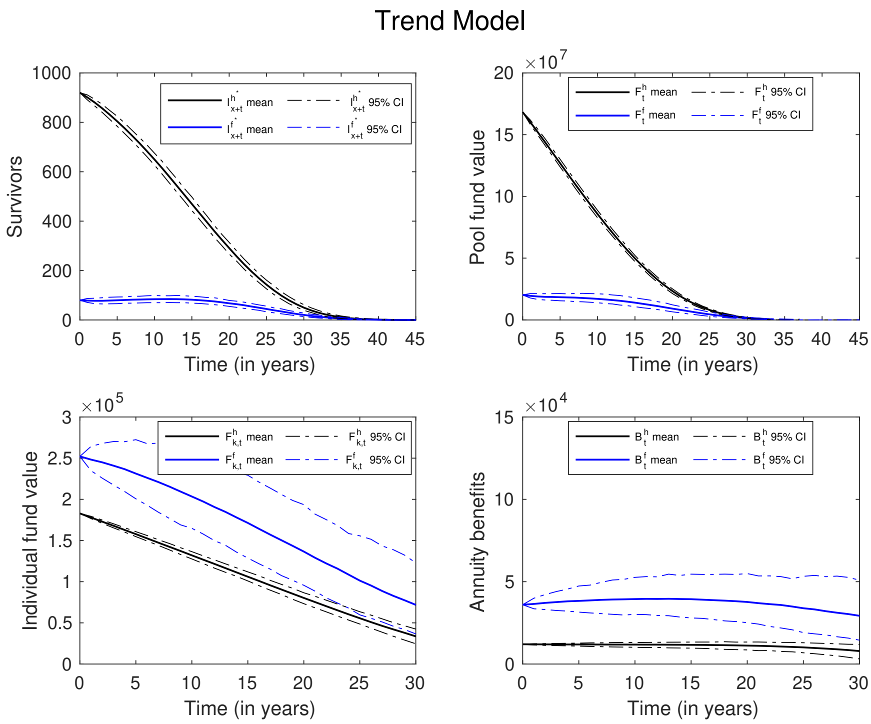


Figure 8: The trend model's number of survivors, pool fund value in healthy and functional disability states, individual fund value and annuity benefits for healthy and disabled participants. The annuity values are computed using the static model which does not include trends and systematic uncertainties, while the dynamics of the fund values are calculated using the trend model assumptions.

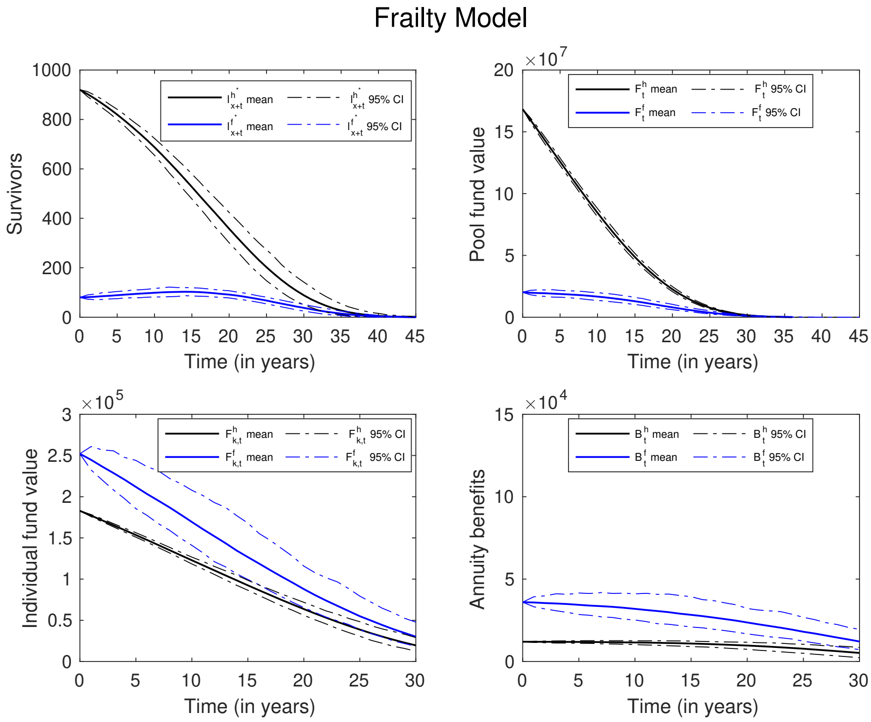


Figure 9: The frailty model's number of survivors in healthy and functional disability states, pool fund value, individual fund value and annuity benefits for healthy and disabled participants. The annuity values are computed using the static model assumptions, while the dynamics of the fund values are calculated using the frailty model assumptions.

To show the likely range of annuity payouts at early and later retirement years, Table 1 shows the mean and the 95% confidence intervals of the annuity payouts at ages 75 and 95 when the time trends and systematic uncertainty are not considered in pricing. We note that the annuity benefits of the healthy participants decline significantly at age 95, as shown by the 5% percentile annuity benefits for the trend and frailty models.

Table 1 also presents annuity payouts when annuity factors are updated with trends and frailty factors, respectively. When systematic trends and uncertainty are considered in pooled health care annuity pricing, future benefit payments for both groups are less susceptible to reductions. However, compared to the initial benefits, the healthy group still receives a slightly lower benefit since their average payments are lower due to less mortality credits than disabled participants.

3.1.2 Numerical example 2

We numerically examine the set up of a multi-state pool using the five-state functional disability and health status model displayed in Figure 3. As in Example 1, we set the initial pool size to 1,000 individuals. Also, based on the HRS data for the interview year 2017, we divide the pool into 650 individuals in good health not functionally disabled, 250 individuals in ill health not functionally disabled, 40 individuals in good health and functionally disabled, and 60 individuals in ill health and functionally disabled.

The initial annuity payout to a healthy pool participant is set to $B_0^h = \$12,000$ per year. For the LTC benefits, we set $c_1 = 1$ to provide $B_0^m = \$24,000$ per year to a participant in ill health not functionally

Table 1: Pooled annuity payments based on the three-state functional disability model with and without systematic trends and uncertainty in pricing from the static, trend and frailty models.

		Age 75			Age 95	
	5%	Mean	95%	5%	Mean	95%
The static model						
Healthy	10,508	11,921	13,094	2,925	11,208	18,648
Disabled	28,258	36,710	47,172	18,400	42,563	88,704
The trend model, without trends in pricing						
Healthy	10,155	11,802	13,066	3,094	7,870	11,945
Disabled	30,153	39,440	52,618	14,496	29,296	50,915
The trend model, with trends in pricing						
Healthy	10,242	11,874	13,149	3,518	11,055	17,407
Disabled	28,533	37,108	49,549	20,013	42,672	75,003
The frailty model, without trends and uncertainty in pricing						
Healthy	10,337	11,619	12,699	2,442	5,189	8,764
Disabled	24,502	31,463	40,253	7,203	12,219	19,475
Frailty model, with trends and uncertainty in pricing						
Healthy	10,346	11,928	13,303	5,314	11,502	16,803
Disabled	28,304	36,539	45,986	23,265	39,174	61,403

disabled, $c_2 = 2$ to provide $B_0^f = \$36,000$ per year to a participant in good health and functionally disabled and $c_3 = 4$ to provide $B_0^{mf} = \$48,000$ per year to a participant in ill health and functionally disabled (see Equation (2.43)). Consistent with Sherris and Wei (2021), the LTC annuity prices are higher for those in ill health since they spend more time disabled. The interest rate is set to $r = 3\%$. The realized interest rate R_t is the same as the expected rate r.

As with the three-state model, we make several assumptions in determining the annuity payments. From the static model, we use the estimated transition probabilities of the static model for determining the annuity values and changes in the realized experience. We present two scenarios for the trend (frailty) model. In the first scenario, the annuity values are computed using the expected one-year transition probabilities matrix of the static model, while the evolution of the fund value is determined using the results from the trend (frailty) model. In the second scenario, the annuity values are computed using estimated transition probabilities from the trend (frailty) model to also include expected improvement in mortality and morbidity in pricing.

Figure 10 shows the mean and the 95% confidence intervals of the number of participants in four different health states from the static model. The expected number of survivors in four different health states is displayed in the top left panel of Figure 10.

The number of participants in both groups decreases with time as many individuals die at older ages. However, the healthy group declines more sharply over the years than the other groups since the number of participants in good health decreases more rapidly as many individuals enter the ill and disability states. The number of sick participants rises initially and gradually decreases since the chronic illness rates increase with age. Also, at younger ages, more participants become sick, and as time progress, many participants in the group die. The number of disabled participants in good and ill health is relatively small and declines more slowly than in the other groups. This is also due to the smaller pool size at the entry time. The total pool size decreases below the threshold from time 23; thus, the plan stops since there are few participants to keep operating the pool and the pool fund value depletes to zero.

On average, the fund value for the healthy group is higher than that of participants in ill health as this group has many participants at the entry time. In contrast, the corresponding individual's value for the healthy participant is relatively lower, followed by that of a sick member, as shown in the bottom left panel of Figure

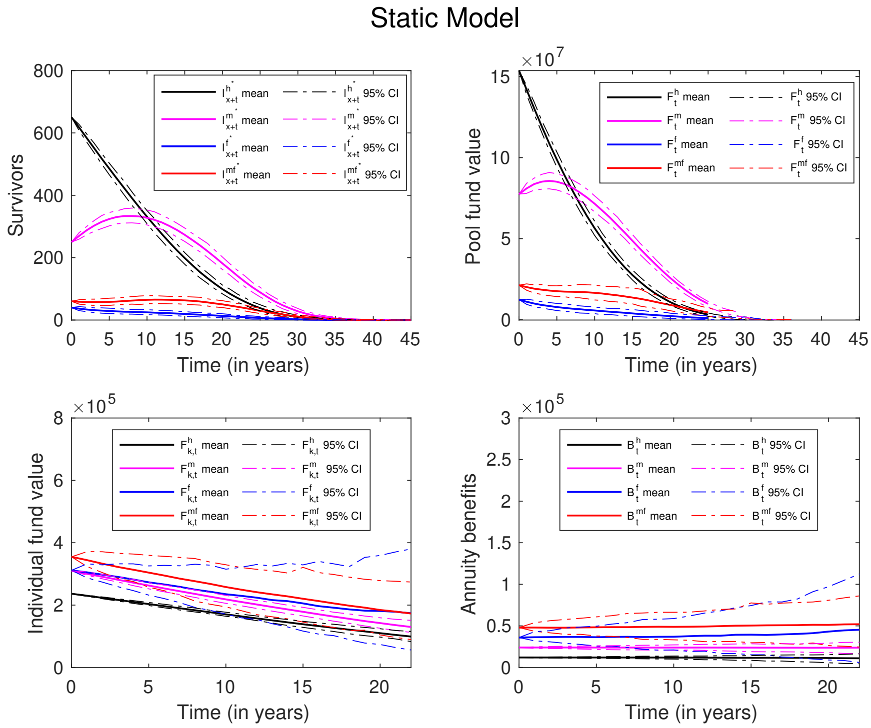


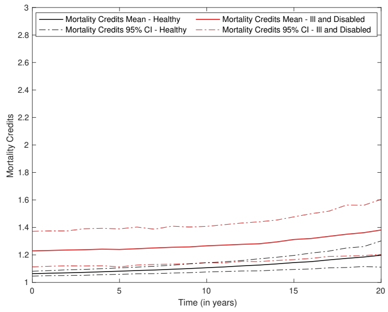
Figure 10: The static model's number of survivors, pool fund value, individual fund value and annuity benefits for participants in four different health states. The annuity values are computed using the static model which does not include trends and systematic uncertainties.

10. These differences are due to varying pool sizes and the annuity payouts throughout the retirement period. For instance, there are increased future benefits for the functionally disabled participants in good and ill health over retirement than the healthy participants. Thus, the functionally disabled individual's fund values are higher than those of the healthy participants.

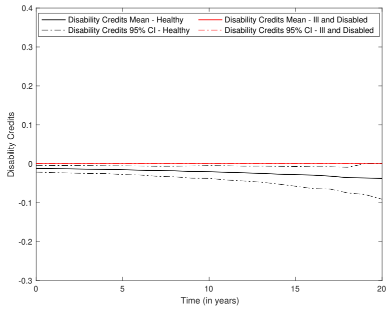
In terms of the annuity benefits, there are increasing benefits for the functionally disabled participants in good and poor health, as shown in the bottom right panel of Figure 10. Especially at more advanced ages, the mean and the 95% quantiles of the functionally disabled in good health are higher than all other groups. On the other hand, healthy and sick participants who are not functionally disabled receive lower but more stable annuity payouts throughout their retirement. Again, the differences in the annuity payments among the groups are due to the differences in mortality, disability and chronic illness credits and group sizes.

To demonstrate the likely difference between annuity benefits for individuals in good health and those in poor health, the values of mortality, disability, and chronic illness credits are displayed in Figure 11. We use the static model results and present values for two groups: healthy and sick participants with functional disabilities. We note that mortality credits for both groups increase over time as the number of deaths increases with age (see Subfigure 11(a)). However, participants in ill health and with functional disability receive higher mortality credits than healthy participants due to higher mortality rates among the sick members.

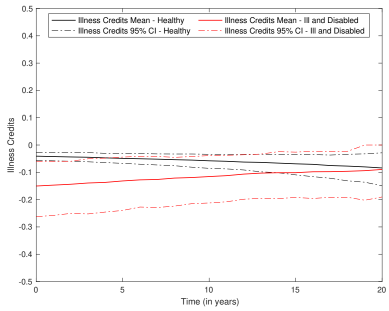
As shown in Subfigure 11(b), there are no disability credits for sick and disabled participants as there are no fund adjustments; an assumption adopted in this paper of no recovery from chronic illness. Thus, a participant who is ill and disabled cannot recover from chronic disease and becomes only functionally



(a) Mortality credits



(b) Disability credits



(c) Illness credits

Figure 11: Mortality, disability and chronic illness credits over time for the healthy, sick and functionally disabled participants based on five-state pooling structure using the functional disability and health status static model.

disabled. On the other hand, healthy participants have disability credits, decreasing over time as more people become functionally disabled at older ages. Since the disability benefits are more expensive, the healthy fund value is credited to fund the increased annuity benefits in dependency. In most cases, sick and disabled participants receive less illness credits than healthy participants, which increases with time since many become ill and disabled as they age. The healthy participants receive higher illness credits, but the trend is downward (see Subfigure 11(c)).

The differences in benefits between participants due to differences in mortality, disability, and chronic illness credits are shown in Table 2. At ages 75 and 80, particularly for the static model, the mean annuity benefits for the healthy participants are lower, followed by those of the sick and disabled members in good health. In contrast, the ill and disabled participants receive higher mean annuity payouts. The volatility in benefits is observed most in the disabled group with good health, as shown by the widest 95% confidence interval. Due to its smaller pool size, the benefits may decrease significantly, but there is also a chance of receiving the highest annuity payout, for example, at age 80.

Figure 12 shows the results from the trend model when the time trend is not included in pricing. Disregarding future expected mortality and morbidity improvement when pricing significantly decreases the annuity benefits, especially for the healthy and sick survivors, as shown in Table 2. In contrast, the disabled participants in good health receive increased annuity payouts at older ages since the static model used in pricing overestimated the mortality and disability rates, ignoring any possibilities for mortality improvement and morbidity compression. The static model is estimated using transition rates, which ignores the impact of any differences between the expected and realized mortality and morbidity experience. If future mortality improvement and morbidity compression are ignored, the annuity prices are underestimated, which increases the realized benefit amounts. The impact of neglecting future trends in mortality and morbidity risk is more significant among healthy members and has a negative effect on annuity payouts. As shown in Table 2 for the trend model, the mean annuity payments at older ages are lower, and the 5% percentile of benefit payments for healthy pool members is very low in comparison to other groups. On the other hand, disabled participants receive higher benefits due to estimation errors. Intuitively, if future morbidity compression is ignored, the insurer will overestimate the prices of the LTC policies.

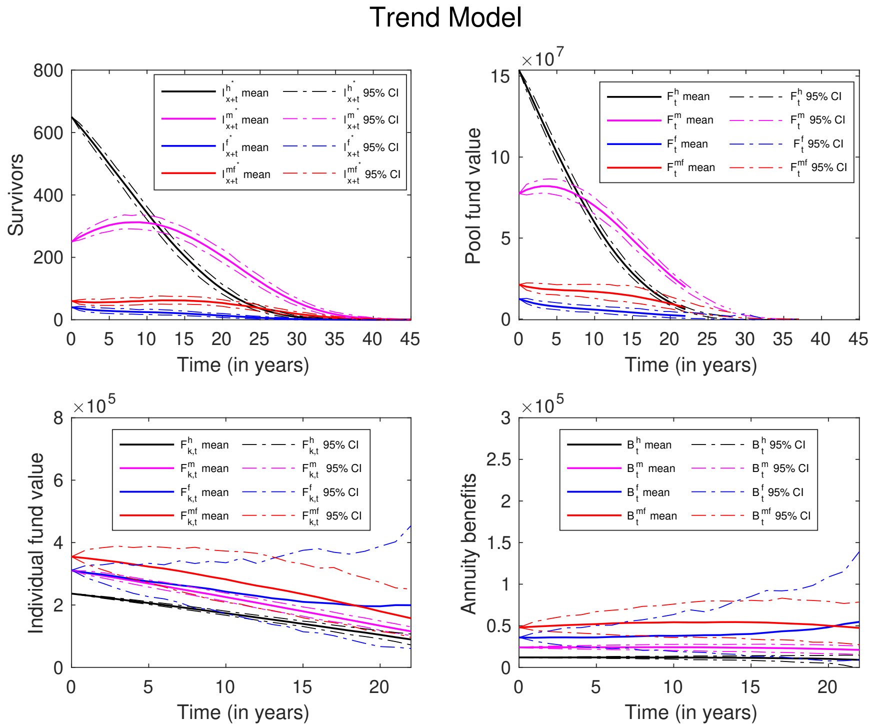


Figure 12: The trend model's number of survivors, pool fund value, individual fund value and annuity benefits for participants in four different health states. The annuity values are computed using the static model which does not include trends and systematic uncertainties and the fund values are computed using trend model.

From the frailty model, Figure 13 shows the results when both time trend and systematic uncertainty are ignored in pricing. As shown in Table 2, the annuity benefits further decline for both groups and the healthy members receive lower annuity benefits than other participants. We also notice that the pool fund value depletes earlier than in the previous cases, and as a result, both groups receive decreased annuity benefits at the reference ages. By contrast, disabled members receive slightly lower annuity benefits than in the previous case, but these still increase with time. Similarly, the static model used in pricing overestimated mortality and disability rates, ignoring any possibilities for mortality improvement and morbidity compression.

We also investigate the impact of including trend and frailty factors in pricing, and the results are shown in Table 2. Incorporating trends in pricing reduces the significance of declining benefits, especially at old ages; for example, with time trends and frailty factors, the mean annuity benefits for the healthy participant increase by 30.3% at age 80, while for the ill participant increase by 19.5% and the disabled in poor health increases by 39.7%. In contrast, annuity benefits for the disabled in good health decrease by 10.5% under the frailty model.

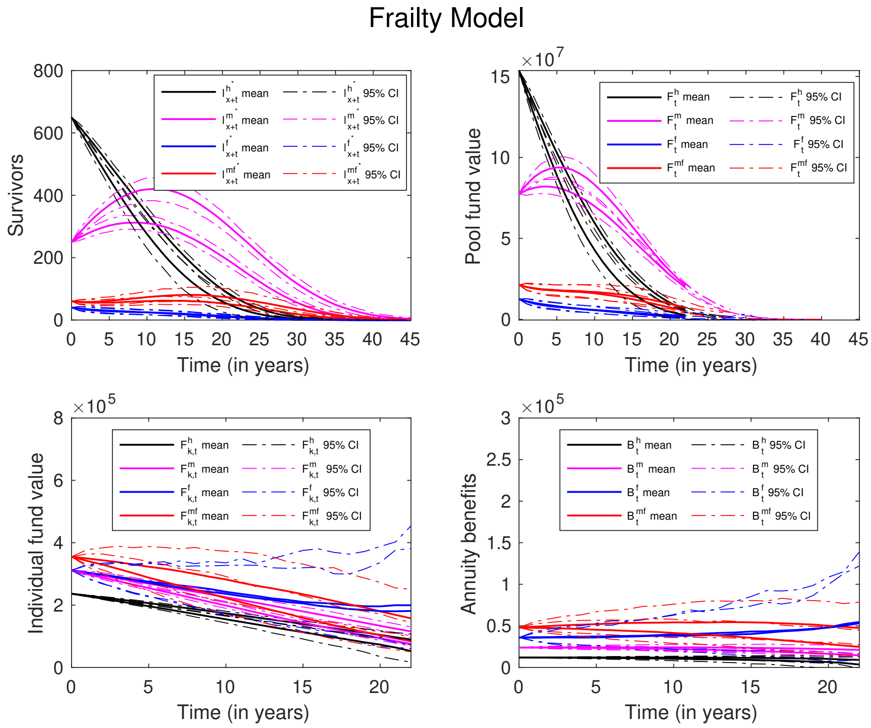


Figure 13: The frailty model's number of survivors, pool fund value, individual fund value and annuity benefits for participants in four different health states. The annuity values are computed using the static model assumptions and the fund values are computed using frailty model results.

These findings show that incorporating both trend and frailty factors when calculating the annuity values reduces the significance of declining benefits and corrects estimation errors in disabled annuity benefits. When there is a trend, it needs to be incorporated into annuity pricing and pooling. If it is not included, it will impact payments as the experience unfolds since it is not anticipated in the pricing. It is important, not critical, that the pricing basis be consistent with the pooling model. The frailty model is the best estimate model capturing expected future trends and systematic uncertainties.

3.2 Comparisons with standard life care

We assess our pooling framework by comparing the present value of the pooled annuity payments with the standard life care product. We estimate the present values of the future annuity payouts from the two designs: three-state functional disability model and five-state functional disability and health status model. The present value (PV) takes the benefit amount at each time t, conditional on survival at time t including all possible transitions, and discounts it to time 0 as given below.

The annuity benefits are the simulated values of the pooled health care annuity product for the static, trend and frailty models. We use the same expressions to calculate the PVs of a standard life care product for the three-state and five-state models. However, since the standard life care product offers a fixed amount of benefits over the retirement period, we use the initial specified benefits at each age over future years in determining the PVs. We also assume zero loadings since the prices are determined based on the best-estimated transition probabilities. In both settings, the values are discounted at the fixed interest rate of 3%. The results from the three-state and five-state models are shown in Tables 3 and 4, respectively.

Table 2: Pooled annuity payments based on the five-state functional disability model with and without systematic trends and uncertainty in pricing from the static, trend and frailty models.

	Age 75			Age 80		
	5%	Mean	95%	5%	Mean	95%
The static model						
Healthy	10,104	11,870	13,780	8,649	11,815	14,617
Ill	20,636	23,818	26,866	19,344	23,677	27,479
Disabled	18,819	38,565	63,829	14,897	39,079	72,339
Ill and Disabled	35,054	49,032	66,469	33, 694	49,978	71,258
The trend model, without trends in pricing						
Healthy	9,905	11,858	13,459	8,343	11,494	14,034
Ill	20,447	24,143	27,656	18,918	23,569	27,790
Disabled	19,198	37,894	63,559	13,222	40,197	85,265
Ill and Disabled	35,899	54,268	76,040	33,918	54,229	80,582
The trend model, with trends in pricing						
Healthy	9,823	11,728	13,374	8,388	11,578	14,198
Ill	20,358	23,881	27,253	18, 809	23,685	27,823
Disabled	19,140	38,380	63,526	13,328	40,001	85,456
Ill and Disabled	32,917	50,190	70,841	31,496	51, 639	77,427
The frailty model, without trends and uncertainty in pricing						
Healthy	8,135	10,500	12,654	4,405	9,010	12,405
Ill	19,569	22,134	24,409	17,363	19,893	22,197
Disabled	21,467	39,755	62,811	20,153	43,219	74,041
Ill and Disabled	28,827	41,438	57,977	22,855	35,567	54,190
The frailty model, with trends and uncertainty in pricing						
Healthy	9,511	11,909	14,135	7,691	11,740	15,351
Ill	20,336	23,754	26,372	19,705	23,771	26,886
Disabled	17,026	37,412	60,535	10,744	38,676	72,294
Ill and Disabled	35,666	49,481	68,189	33,360	49,689	72,792

Table 3: Present values of the annuity payments using the three-state functional disability model from the static, trend and frailty models.

	Standard Life Care ($)			Pooled Health Care ($)		
	5%	Mean	95%	5%	Mean	95%
Using the static model						
Healthy	182,024	182,024	182,024	178,073	182,634	187,635
Disabled	251,527	251,527	251,527	236,131	253,100	273,433
Using the trend model						
Healthy	183,404	183,404	183,404	179,029	183,911	188,716
Disabled	248,525	248,525	248,525	233,447	250,325	272,741
Using the frailty model						
Healthy	197,297	203,263	209,110	195,537	203,581	210,913
Disabled	271,641	273,944	276,219	258,299	275,194	296,214

The static and trend models have deterministic values in transition probabilities and annuity benefits; thus, the mean PVs and the corresponding 95% confidence interval of the standard life care product are constant. On the other hand, the values from the frailty model vary, demonstrating the model's variability as it includes a stochastic component in pricing. The PVs of the pooled health care annuity benefits exhibit variability for both static, trend, and frailty models obtained from the simulated outcomes at each age.

Table 4: Present values of the annuity payments using the five-state functional disability and health status model from the static, trend and frailty models.

	Standard Life Care ($)			**Pooled Health Care ($)**		
	5%	Mean	95%	5%	Mean	95%
Using the static model						
Healthy	213,621	213,621	213,621	208,541	213,972	219,131
Ill	295,002	295,002	295,002	285,629	295,397	306,472
Disabled	276,611	276,611	276,611	254,643	279,517	310,529
Ill and disabled	331,753	331,753	331,753	307,331	333,267	366,296
Using the trend model						
Healthy	213,048	213,048	213,048	207,650	213,206	218,707
Ill	291,886	291,886	291,886	281,364	292,757	304,297
Disabled	274,178	274,178	274,178	254,773	278,717	310,301
Ill and disabled	322,933	322,933	322,933	298,439	324,960	359,510
Using the frailty model						
Healthy	242,358	252,353	262,680	241,482	252,649	264,531
Ill	327,791	332,372	337,260	321,629	332,985	345,643
Disabled	315,994	324,657	333,701	297,856	327,323	363,077
Ill and disabled	373,212	382,996	393,787	353,111	385,824	424,304

In both settings, the mean PVs of the pooled health care product are the same as that of the standard life care, implying that the proposed plan is actuarial fair. The slight difference in mean PVs for individuals in worse health conditions (disabled for the three-state model and sick and disabled participants for the five-state model) is due to the smaller pool sizes. In the numerical analysis, for example, for the five-state model, we assumed 40 sick and disabled participants at the start; this smaller group size causes slight deviations in the mean PV of the pooled health care annuity benefits. However, for a sufficiently large pool, the results of the mean PV are also the same for both products.

In addition, the PVs of the annuity benefits for the healthy group are much lower than other participants for both products. Also, the static model has a much lower premium for the healthy, sick and disabled groups than the trend and frailty models. It underestimates the premiums by not considering future mortality improvement and morbidity compression. Another interesting observation is that the uncertainty in future benefits is much reduced under the frailty model than in trend and static models. This is consistent with the existing literature such as Sherris and Wei (2021).

3.3 Changes in the pool size

We increase the size of the initial pool to 10,000 people and use the same ratios to determine the number of participants in different health states. For the three-state model, we assume 9,200 healthy participants and 800 disabled. For the five-state model, there are 6,500 participants in good health without functional disability, 2,500 participants in ill health without functional disability, 400 participants in good health with functional disability and 600 participants in ill health with functional disability. Similarly, we implement two scenarios for larger pool sizes. The first scenario shows no expected improvement and systematic uncertainty in the annuity factors, while in the second one, the annuity factors are updated to consider trends and systematic uncertainty. Table 5 shows the mean payments and the 95% confidence intervals for the healthy and disabled participants at age 75 and 95. We note that as the pool size increases, the threshold time to keep operating the pool increases. However, regardless of pool size, average future benefit payments decrease for the healthy group and increase for the disabled, especially when systematic trends and pricing uncertainty are excluded in pricing, as shown in Table 5.

Table 5: Pooled annuity payments without systematic trends and uncertainty using the three-state functional disability static, trend and frailty models for a larger pool size.

	Age 75			Age 95		
	5%	Mean	95%	5%	Mean	95%
The static model						
Healthy	11,540	11,998	12,429	9,546	11,852	13,871
Disabled	33,151	36,017	39,319	28,938	36,897	47,408
The trend model, without trends in pricing						
Healthy	11,438	11,919	12,370	7,089	8,405	9,645
Disabled	35,141	38,389	41,930	20,454	25,154	30,815
The trend model, with trends in pricing						
Healthy	11,528	11,992	12,429	9,914	11,861	13,630
Disabled	33,011	36,111	39,397	29,937	36,683	45,038
The frailty model, without trends and uncertainty in pricing						
Healthy	11,065	11,653	12,226	2,997	5,278	8,186
Disabled	27,862	30,453	33,552	8,461	10,246	12,296
The frailty model, with trends and uncertainty in pricing						
Healthy	11,525	11,984	12,423	10,426	11,961	13,522
Disabled	33,327	36,105	38,942	30,889	36,259	42,291

From the three-state trend model, the average benefits for the disabled first increase before significantly declining at older ages when expected improvements are not included in pricing. In contrast, the mean payments for healthy participants decrease throughout retirement (see Table 5). With allowance for future expected improvements in mortality and morbidity risks, the annuity benefits do not decline significantly at old age. However, increasing the pool size slightly decreases the mean and the 5% percentile of benefit payments for the healthy and disabled participants at ages 75 and 95, compared to the small pool size. It also decreases the absolute volatility in payments.

Furthermore, the three-state frailty model shows that even with a larger pool of 10,000 participants, annuity benefits drop significantly when trends and systematic uncertainty are not factored into pricing. It is therefore critical to include trends and systematic uncertainty to reduce benefit volatility regardless of pool size. Similarly, for the five-state model, when trends and systematic uncertainty are not factored into pricing, annuity benefits decline significantly, even for larger pool sizes (as shown in Table 6). On the other hand, a larger fund with 10,000 initial participants has slightly narrower confidence intervals than a smaller fund with 1,000 initial participants. When the pool size is increased, the idiosyncratic variability is significantly reduced. Annuity benefits for all other groups, except the functionally disabled in good health, decrease with age regardless of pool size.

3.4 Discussion

We have presented two examples of how individuals in different health states can share mortality and health risks in a pooled annuity fund. The first example examines the three-state functional disability model from the point of view of mortality and disability risks. In contrast, the second considers the five-state functional disability and health status model from the perspective of mortality, disability, and chronic illness risks. The impact of pooling various health risks on future benefits is determined by the size of the initial pool and the inclusion of systematic trends and uncertainty in pricing. The plan operates longer as the initial pool size increases and when systematic trends and uncertainty are included in pricing. The annuity benefits for the healthy members slightly increase at the reference ages (ages 75 and 95 for the three-state model and ages 75 and 80 for the five-state model). In contrast, with 10,000 initial participants, the disabled and sick receive slightly reduced mean annuity benefits at the reference ages. On the other hand, the idiosyncratic

Table 6: Pooled annuity payments without systematic trends and uncertainty using the five-state functional disability and health status static, trend and frailty models for a larger pool size.

	Age 75			Age 95		
	5%	Mean	95%	5%	Mean	95%
The static Model						
Healthy	11,386	11,950	12,392	10,593	11,917	13,115
Ill	23,126	24,029	24,964	22,414	24,002	25,650
Disabled	31,395	36,387	42,394	27,061	36,677	47,744
Ill and disabled	43,402	47,971	52, 273	41,158	48,192	55,503
The trend model, without trends in pricing						
Healthy	11,539	12,100	12,579	9,806	11,169	12,382
Ill	22,992	24,262	25,317	21,2 29	22,585	23,936
Disabled	30,304	35,860	43,185	28,683	39,257	50,898
Ill and disabled	46,028	51,895	58,598	38,597	44,945	53,301
The trend model, with trends in pricing						
Healthy	11,365	11,961	12,505	10,355	11,916	13,340
Ill	22,841	24,033	25,060	22,130	23,952	25,662
Disabled	31,152	36,545	42,654	26,450	36,919	48,321
Ill and Disabled	42,934	47,847	54,452	40,675	48,108	56,685
The frailty model, without trends and uncertainty in pricing						
Healthy	9,465	10,691	11,833	1,219	7,423	11,427
Ill	21,476	22,383	23,179	15,360	16,583	17,534
Disabled	32,047	37,753	44,498	31,004	41,340	56,112
Ill and disabled	34,742	39,533	45,696	18,709	25,738	36,862
The frailty model, with trends and uncertainty in pricing						
Healthy	11,236	12,000	12,699	9,837	11,912	13,589
Ill	23,194	23,954	24,680	22,672	23,900	24,970
Disabled	30,643	36,525	43,234	26,101	37,444	49,786
Ill and Disabled	43,863	48,149	52,558	42,407	48,507	56,153

variability is significantly reduced. A larger fund with 10,000 initial participants has slightly narrower confidence intervals in annuity benefits, even at old ages, than a smaller fund with 1,000 initial participants. The wide range of benefit outcomes at the older ages remains, along with the reduction in average benefits.

Regardless of the number of people in the pool, the expected annuity benefits for the healthy group are always less than the initially specified benefit amounts. This is due to lower mortality, disability, and chronic illness credits within the group. Furthermore, as people age, the number of people in the health group decreases significantly since many individuals transition to poorer health states, reducing mortality and disability credits. Annuity payouts can be improved even further by considering significant equity investment at the expense of volatility. The pooled health care annuity benefits for individuals in poor health states tend to increase over time, especially when trends and systematic uncertainty are incorporated in pricing. The smaller pool sizes and the higher mortality credits for these participants are the two main factors for the increased benefits in dependency. With an increase in the pool size, we note a slight reduction in the benefits for individuals in poor health states. Even with significant-sized pools, one factor that clearly undermines the effectiveness of the pooled annuity benefits is the presence of systematic trends and uncertainty in mortality. The annuity benefits decline at older ages, even for the disabled participants reflecting the impact of systematic improvement in mortality rates (especially for the three-state model as shown in Table 5). The downside is concerning as the 5% percentile at older ages represents a significant reduction in benefit payments, particularly for healthy participants.

With frailty and trend factors, the range in the annuity benefits is narrower, reflecting the decreasing volatility of future benefits in the pool, and the benefits do not drop significantly. We conclude that pooling different groups in the presence of systematic trends and uncertainty improves annuity benefits, particularly for people in poor health. For example, in the three-state model, annuity benefits are relatively higher for individuals with disabilities, whereas in the five-state model, annuity benefits are higher for participants in ill health and with disability. Approaches to annuity factors estimation that does not include systematic trends and uncertainty in mortality and morbidity risk result in higher reductions in annuity payments in later years. This affects all other groups except the functionally disabled. The annuity benefits significantly decline for the healthy group compared to the sick members since the expected improvement in mortality in the healthy group is more significant, thus, reducing the benefits. The static model tends to overestimate disability and mortality rates for the functionally disabled, while the estimated models with trends show that the elderly have a lower probability of being disabled. This is why the annuity values with no expected improvements in morbidity risk are lower, which tends to increase disability payments.

4 Conclusion and Future Research

The increasing demand for LTC annuities has motivated us to explore innovative designs in pooled annuity products, including long-term care insurance. We propose a pooled annuity concept for heterogeneous individuals classified according to functional disability states and chronic illness status. We use multi-state models calibrated to US data to assess the impact of including systematic trends and uncertainty, along with recovery possibilities, on pooled health care annuity payments. The results show that incorporating trends and frailty factors in pooling differing risks improves the annuity payouts, especially for those in poor health, reflecting actual experience trends results in a better annuity experience for pool members. When pricing the pooled annuity products, we emphasize that individuals must be distinguished according to different risks, which improves the pooled annuity products' value in terms of higher mean annual payouts, lower volatility, and lower downside risk.

We focus on mortality, morbidity and health risk factors and assume risk free interest rates. These assumptions enable us to illustrate how heterogeneous individuals can share mortality and health risks in a pooled annuity fund and are consistent with traditional life annuities and life care annuities. More importantly, in a pooled arrangement, it is common to share investment risk as well; the participants may wish to invest in equity markets to improve the annuity benefits with volatility management strategies for minimizing the impact of downward market risks. This has been done for pooled annuity products - see for example, Li et al. (2022) and Olivieri et al. (2022); extending this analysis to the broader risk sharing is an interesting topic of future research.

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A Proof of a General Pooling Framework

This is a proof to show that $\mathbf{B}_t = \mathbf{B}_{t+1}$ if $\mathbf{P}^*_{x+t} = \mathbf{P}_{x+t}$ and $R_t = r$.

From

$$\mathbf{F}_t = \mathbf{L}_{x+t} \otimes (\mathbf{A}_{x+t}\mathbf{B}_t) \tag{A.1}$$

given that

$$\mathbf{A}_{x+t} = \mathbf{I} + \upsilon \mathbf{P}_{x+t}\mathbf{A}_{x+t+1}, \tag{A.2}$$

we can write Equation (A.1) as

$$\begin{aligned} \mathbf{F}_t &= \mathbf{L}_{x+t} \otimes (\mathbf{I} + \upsilon \mathbf{P}_{x+t}\mathbf{A}_{x+t+1})\mathbf{B}_t, \\ &= \mathbf{L}_{x+t} \otimes (\mathbf{I}\mathbf{B}_t + \upsilon \mathbf{P}_{x+t}\mathbf{A}_{x+t+1}\mathbf{B}_t), \\ &= \mathbf{L}_{x+t} \otimes \mathbf{I}\mathbf{B}_t + \mathbf{L}_{x+t} \otimes (\upsilon \mathbf{P}_{x+t}\mathbf{A}_{x+t+1}\mathbf{B}_t). \end{aligned} \tag{A.3}$$

Then

$$\mathbf{F}_t - \mathbf{L}_{x+t} \otimes \mathbf{I}\mathbf{B}_t = \mathbf{L}_{x+t} \otimes (\upsilon \mathbf{P}_{x+t}\mathbf{A}_{x+t+1}\mathbf{B}_t), \tag{A.4}$$

but $\upsilon = \frac{1}{1+r}$ hence

$$\mathbf{F}_t - \mathbf{L}_{x+t} \otimes \mathbf{I}\mathbf{B}_t = \mathbf{L}_{x+t} \otimes (\mathbf{P}_{x+t}\mathbf{A}_{x+t+1}\mathbf{B}_t)\frac{1}{1+r}. \tag{A.5}$$

Dividing element-wise by $\mathbf{L}_{x+t}$ on both sides of Equation (A.5), we obtain

$$\frac{\mathbf{F}_t - \mathbf{L}_{x+t} \otimes \mathbf{I}\mathbf{B}_t}{\mathbf{L}_{x+t}} = (\mathbf{P}_{x+t}\mathbf{A}_{x+t+1}\mathbf{B}_t)\frac{1}{1+r}. \tag{A.6}$$

Multiplying by $(1 + R_t)$ on both sides, Equation (A.6) becomes

$$\left(\frac{\mathbf{F}_t - \mathbf{L}_{x+t} \otimes \mathbf{I}\mathbf{B}_t}{\mathbf{L}_{x+t}}\right)(1 + R_t) = (\mathbf{P}_{x+t}\mathbf{A}_{x+t+1}\mathbf{B}_t)\frac{1}{1+r} \cdot (1 + R_t), \tag{A.7}$$

then multiplying by $(\mathbf{P}^*_{x+t})^{-1}$ on both sides, we obtain

$$(\mathbf{P}^*_{x+t})^{-1}\left(\frac{\mathbf{F}_t - \mathbf{L}_{x+t}\otimes \mathbf{I}\mathbf{B}_t}{\mathbf{L}_{x+t}}\right)(1+R_t) = (\mathbf{P}^*_{x+t})^{-1}(\mathbf{P}_{x+t}\mathbf{A}_{x+t+1}\mathbf{B}_t)\frac{1}{1+r}\cdot(1+R_t). \tag{A.8}$$

If $\mathbf{P}^*_{x+t} = \mathbf{P}_{x+t}$ then $(\mathbf{P}^*_{x+t})^{-1}\mathbf{P}_{x+t} = \mathbf{I}$ and if $R_t = r$ then $\frac{1+R_t}{1+r} = 1$.
Thus Equation (A.8) becomes

$$(\mathbf{P}^*_{x+t})^{-1}\left(\frac{\mathbf{F}_t - \mathbf{L}_{x+t}\otimes \mathbf{I}\mathbf{B}_t}{\mathbf{L}_{x+t}}\right)(1+R_t) = \mathbf{A}_{x+t+1}\mathbf{B}_t. \tag{A.9}$$

By element-wise multiplying $\mathbf{L}_{x+t+1}$ to Equation (A.9), we obtain

$$\mathbf{L}_{x+t+1}\otimes\left((\mathbf{P}^*_{x+t})^{-1}\left(\frac{\mathbf{F}_t - \mathbf{L}_{x+t}\otimes \mathbf{I}\mathbf{B}_t}{\mathbf{L}_{x+t}}\right)(1+R_t)\right) = \mathbf{L}_{x+t+1}\otimes(\mathbf{A}_{x+t+1}\mathbf{B}_t). \tag{A.10}$$

Based on Equation (2.15), we can write Equation (A.10) as

$$\mathbf{F}_{t+1} = \mathbf{L}_{x+t+1}\otimes(\mathbf{A}_{x+t+1}\mathbf{B}_t), \tag{A.11}$$

and by determining $\mathbf{B}_{t+1}$ from Equation (A.11) we can show that

$$\mathbf{B}_{t+1} = \mathbf{B}_t. \tag{A.12}$$